THE

MATHEMATICAL GAZETTE.

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LOCAL BRANCHES.

THE LONDON BRANCH.

A meeting of the Branch was held at the London Day Training College, on Saturday, March 8th. The following papers were read:

"The Parallel Axiom," by the Rev. W. B. Frankland.

"The Teaching of Elementary Geometry and Trigonometry," by Mr. W. J. Dobbs.

In the unavoidable absence of Mr. Frankland, his paper was read by Mr. F. C. Boon. An interesting discussion followed.

NORTH WALES BRANCH.

A meeting of this Branch was held on Feb. 22 at the Bangor County School for Girls. Professor Bryan opened a discussion, in which nearly all the members present took part, on the unsuitability of some of the mathematical questions set at school examinations, several instances being quoted from recent papers. The better questions were considered to be those which involved physical and dynamical quantities. It was decided to recommend the Central Welsh Board to include an elementary mechanics paper in their additional mathematics examination.

UNIVERSITY OF EDINBURGH.

MATHEMATICAL LABORATORY.

The Court of the University of Edinburgh have resolved upon the establishment of a Laboratory for practical instruction in numerical, graphical, and mechanical calculation and analysis, as required in the applied mathematical sciences, and for research in connection with the Mathematical Department.

The Laboratory will be opened in October of the present year (1913), under the direction of Professor E. T. Whittaker, Sc.D., F.R.S., and the Lecturers in the Mathematical Department. The nature of the course will be understood from the following schedule:

Course of Practical Work in the Mathematical Laboratory.

Differences and interpolation: computations with tables of logarithms, log sines, natural sines, products, quarter-squares, etc.: numerical solution of trigonometric problems.

Controls for checking accuracy of computations: design of computing-forms.

Method of least squares: numerical solutions of systems of linear equations: numerical evaluation of determinants.

Curve-fitting. Calculation of correlation-coefficients.

Analysis of a function into sine and cosine terms (practical Fourier analysis).

Analysis for the discovery of periodic constituents in a function (periodogram analysis). Practical spherical harmonic analysis. Other methods of analysis of functions empirically given.

Construction of curves and surfaces: linkages: roulettes. Projections: photogrammetry: map-making. Graphic solution of numerical equations: graphic and mechanical solution of problems in spherical trigonometry: nomography. Applications of triangle of vectors.

Use of instruments employed in calculation, especially slide rules, arithmometers, planimeters, integraphs, and harmonic analysers.

Numerical evaluation of definite integrals.

Numerical solution of differential equations.

Numerical evaluation of roots, etc., of transcendental functions.

*Calculations performed with elliptic functions: arcs on spheroids, etc.

*Formation and use of tables of Legendre's and Bessel's functions, the Gamma function, Error-function, and other transcendental functions.

*Construction of tables of new functions, and functions not previously tabulated, including automorphic functions, and the parabolic-cylinder and elliptic-cylinder functions.

Facilities will be given for original research in these subjects. In this connection, attention may be drawn to the aid afforded by the extensive tables of E. Sang, comprising 47 MSS. volumes, which are deposited in Edinburgh.

Persons (other than present students of Edinburgh University) who wish to take a course in the Laboratory, or to engage in original research in connection with it, are requested to communicate with Professor Whittaker, 35 George Square, Edinburgh, as soon as possible. The University grants recognition, under certain conditions, to Research Students, who are permitted to offer themselves for the degree of D.Sc.

April 1913.

THE NAPIER TERCENTENARY CELEBRATION.

Some years ago the Misses Sang, daughters of the late Dr. Edward Sang, gave to the Scottish nation the remarkable collection of their father's manuscript volumes of logarithmic and trigonometrical tables, in which the values were calculated accurately to fifteen significant figures. The thirty large quarto volumes were entrusted to the care of the Council of the Royal Society of Edinburgh, who were given full powers to publish any or all of the tables, should such publication be found advisable and possible. The Royal Society Committee in charge of the negotiations which led to this important transaction, having their attention drawn to the history of the discovery of logarithms, suggested to the Council of the Royal Society that a Napier Celebration be held commemorating the tercentenary of the publication in 1614 of Napier's first book of logarithms. This was agreed to; and towards the end of last year the Council invited various educational and scientific corporations and societies to appoint delegates to a General Committee to discuss the proposal.

*Sufficient theoretical explanation will be given to render this part of the course intelligible to those who have not previously studied the functions of higher analysis.

In the first instance only Scottish institutions were approached, with the exception of the Royal Society of London and the Royal Astronomical Society, the former on account of its pre-eminence in the world of science, the latter because Napier's invention was of special value to astronomers, and was indeed first fully appreciated by them.

This Committee, which included such well-known names as those of Dr. Glaisher, Major Macmahon, Professor H. M. Macdonald, Professor Sampson, Professor Gibson, Professor Gray, Professor Whittaker, Professor Steggall, met in the Royal Society Rooms, 22 George Street, Edinburgh, on Saturday, Feb. 22, at 11 a.m.

Mr. J. R. Findlay, one of the representatives of the Edinburgh Merchant Company, was voted to the Chair, and Dr. C. G. Knott, General Secretary of the Royal Society, acted as secretary.

Among the apologies for absence was a letter from Dr. J. W. L. Glaisher, one of the representatives of the Royal Astronomical Society. A few sentences are well worth quotation. After explaining why he was not able to come to the meeting, Dr. Glaisher wrote: "More than forty years ago when I wrote my British Association report on Mathematical Tables I was *intensely* impressed by the splendid effort of mind by which logarithms were invented, and the intervening years have only heightened this feeling. I have always felt that next to Newton's proof of universal gravitation (and his showing that planets move under it in conic sections) Napier's invention is the greatest contribution to science which has ever been made in these isles (of course I omit recent scientific discoveries about which it is too early to judge), and I always regard Napier as second only to Newton, and a close second. . . . I am very glad the memorial is to take place, for Napier's work was simply transcendent, and the publication of the little quarto work *does* deserve to be commemorated. Of course I will do anything I can. Anyway I am very glad to be connected with it, and so indirectly with Napier. I never get over the *romance* of the invention, *i.e.* that it took place *when* it did, and where it did—so much before it was 'due' mathematically—from the train of thought which led to it."

This letter gave the meeting a splendid start; and as the discussion developed it was quite clear that the members of the Committee were practically agreed to hold a really adequate commemoration of one of the greatest men Scotland had ever produced. This was not merely true of those who were directly interested in mathematics, but also of the others whose main interests were not scientific. With admirable perspicuity, the Chairman, who is one of the best known citizens of Edinburgh, was soon able to gather up the drift of the discussion into a series of resolutions, which were formally passed by the General Committee and then entrusted to the care of an Executive Committee.

Dr. C. G. Knott (Royal Society of Edinburgh) and Dr. A. E. Sprague (Faculty of Actuaries) were appointed honorary secretaries, and Adam Tait, Esq., Royal Bank of Scotland, the honorary treasurer. The Executive Committee were given power to add to their number, and to appoint sub-committees to take charge of the various lines of work indicated by the resolutions.

As so far developed the celebration is to take the form of a Congress in the summer of 1914, opening with a public reception and an Address by an eminent man upon some aspect of Napier's Life and Work. The old Baronial Castle of Merchiston is now one of the outstanding schools in Edinburgh, and Sir Thomas Clouston, M.D., in the name of the Directors of the Merchiston Castle School, invited the members of the Congress to a Garden Party to be held in the Castle grounds.

Another feature will be the exhibits of all kinds of calculating machines and logarithmic and mathematical tables necessary for calculation. It is proposed also to invite eminent mathematicians from foreign countries to

take part in the celebrations, perhaps contributing important papers on methods of calculation. A memorial volume will be published, containing the addresses and communications which will form the business of the Congress.

To meet necessary expenses a donation list is to be opened, to which societies and individuals may contribute. It is also hoped that many interested in the celebration will become Founder Members, the subscription to be £2. The ordinary membership subscription is to be kept as low as possible. It is expected that these membership subscriptions will cover the expenses of the memorial volume.

Although, as already noted, the General Committee was at first composed of representatives mainly from Scottish societies and institutions, the intention is, now that the project has been fairly started, to ask the co-operation of similar societies throughout the British Empire, whose world-wide extension is intimately bound up with the art of navigation, itself an art greatly facilitated if not really created by the invention of the logarithm.

C. G. K.

THE POWER-SUM FORMULA AND THE BERNOULLIAN FUNCTION.

(*Continued from* Vol. VI. p. 336.)

III. The Euler-Maclaurin Formula.

10. The ordinary formula for Sn^r in terms of n, and the alternative formula in terms of $n+\frac{1}{2}\equiv\nu$, are particular cases of the Euler-Maclaurin formula and of the alternative formula involving "mid-ordinates"; these latter formulae being obtainable from (2) and (3) of § 5, with D and n^r replaced by hD and a polynomial in x.

The formulae are usually regarded by mathematicians as a means of expressing a sum by an integral; but their practical value is for the converse purpose, viz. the expression of an area in terms of ordinates. Taking $f(x)$ to be a polynomial, they can be written

$$\begin{aligned} f(a+nh)-f(a)=h\{&\tfrac{1}{2}f'(a)+f'(a+h)+f'(a+2h)+\ldots\\ &\ldots+f'(a+(n-1)h)+\tfrac{1}{2}f'(a+nh)\}\\ &-B_1/2!\,.\,h^2\{f'''(a+nh)-f'''(a)\}\\ &+B_2/4!\,.\,h^4\{f^{\mathrm{v}}(a+nh)-f^{\mathrm{v}}(a)\}\\ &-\ldots, \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

$$\begin{aligned} f(a+nh)-f(a)=h\{&f'(a+\tfrac{1}{2}h)+f'(a+\tfrac{3}{2}h)+\ldots+f'(a+(n-\tfrac{1}{2})h)\}\\ &+p_1/2!\,.\,h^2\{f'''(a+nh)-f'''(a)\}\\ &-p_2/4!\,.\,h^4\{f^{\mathrm{v}}(a+nh)-f^{\mathrm{v}}(a)\}\\ &+\ldots. \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

The second formula can be obtained from the first by expressing separately $f(a+nh)-f(a)$ and $2\{f(a+2n\,.\,\frac{1}{2}h)-f(a)\}$, and subtracting the one expression from the other.

11. We may regard these formulae as built up by considering successive strips of the figure whose ordinate is $f'(x)$, the fundamental formulae for a strip being

$$\begin{aligned} f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h)=h\,.\,&\tfrac{1}{2}\{f'(x+\tfrac{1}{2}h)+f'(x-\tfrac{1}{2}h)\}\\ &-B_1/2!\,.\,h^2\{f'''(x+\tfrac{1}{2}h)-f'''(x-\tfrac{1}{2}h)\}\\ &+B_2/4!\,.\,h^4\{f^{\mathrm{v}}(x+\tfrac{1}{2}h)-f^{\mathrm{v}}(x-\tfrac{1}{2}h)\}\\ &-\ldots, \end{aligned} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

$$f(x+\tfrac{1}{2}h)-f(x-\tfrac{1}{2}h)=hf'(x)$$
$$+p_1/2!\,.\,h^2\{f'(x+\tfrac{1}{2}h)-f'(x-\tfrac{1}{2}h)\}$$
$$-p_2/4!\,.\,h^4\{f^{\mathrm{iv}}(x+\tfrac{1}{2}h)-f^{\mathrm{iv}}(x-\tfrac{1}{2}h)\}$$
$$+\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20)$$

or, in the central-difference notation,

$$\delta f(x)=h\mu f'(x)-B_1/2!\,.\,h^2\delta f''(x)+B_2/4!\,.\,h^4\delta f^{\mathrm{iv}}(x)-\ldots, \quad \ldots\ldots\ldots(21)$$

$$\delta f(x)=hf'(x)+p_1/2!\,.\,h^2\delta f''(x)-p_2/4!\,.\,h^4\delta f^{\mathrm{iv}}(x)+\ldots. \quad \ldots\ldots\ldots\ldots(22)$$

Regarding $f'(x)$ as a given ordinate, we may say that in the one case our unit is the area from $f'(x)$ to $f'(x+h)$, while in the other case it is the area from $f'(x-\frac{1}{2}h)$ to $f'(x+\frac{1}{2}h)$. But we can obtain a more fundamental formula by dividing each of these strips into two. It will be found that this fundamental formula may be written

$$f(x+\tfrac{1}{2}h)-f(x)=\tfrac{1}{2}hf'(x)$$
$$+h^2\{p_1f''(x+\tfrac{1}{2}h)+B_1f''(x)\}/2!$$
$$-h^4\{p_2f^{\mathrm{iv}}(x+\tfrac{1}{2}h)+B_2f^{\mathrm{iv}}(x)\}/4!$$
$$+h^6\{p_3f^{\mathrm{vi}}(x+\tfrac{1}{2}h)+B_3f^{\mathrm{vi}}(x)\}/6!$$
$$-\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

Changing h into $-h$, we obtain

$$f(x)-f(x-\tfrac{1}{2}h)=\tfrac{1}{2}hf'(x)$$
$$-h^2\{B_1f''(x)+p_1f''(x-\tfrac{1}{2}h)\}/2!$$
$$+h^4\{B_2f^{\mathrm{iv}}(x)+p_2f^{\mathrm{iv}}(x-\tfrac{1}{2}h)\}/4!$$
$$-h^6\{B_3f^{\mathrm{vi}}(x)+p_3f^{\mathrm{vi}}(x-\tfrac{1}{2}h)\}/6!$$
$$+\ldots. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(24)$$

If we add (23) and (24), we obtain (20); while, if we change x into $x-\frac{1}{2}h$ in (23) and into $x+\frac{1}{2}h$ in (24), and then add, we obtain (19).

12. In all the above formulae it is supposed that $f(x)$ is a polynomial in x, and that the series is continued until the derived functions vanish. When this is not the case, the "remainder" has to be considered. The remainder in the ordinary formula (17) is usually given in a form involving the function $\phi(\theta)$ defined by (11); but it will be found that neater results are in all cases obtained by using $\Phi(\theta)$ and $\Psi(\theta)$. The fundamental formula may be taken as corresponding to (23). If we write

$$f(x+\tfrac{1}{2}h)-f(x)=\tfrac{1}{2}hf'(x)$$
$$+h^2\{p_1f''(x+\tfrac{1}{2}h)+B_1f''(x)\}/2!$$
$$-h^4\{p_2f^{\mathrm{iv}}(x+\tfrac{1}{2}h)+B_2f^{\mathrm{iv}}(x)\}/4!$$
$$+h^6\{p_3f^{\mathrm{vi}}(x+\tfrac{1}{2}h)+B_3f^{\mathrm{vi}}(x)\}/6!$$
$$-\ldots$$
$$+(-)^{q-1}h^{2q}\{p_qf^{(2q)}(x+\tfrac{1}{2}h)+B_qf^{(2q)}(x)\}/(2q)!$$
$$+Q_q, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25)$$

then

$$Q_q=h^{2q+1}/(2q)!\,.\int_0^{\frac{1}{2}}\Phi_{2q}(\theta)f^{(2q+1)}(x+\theta h)d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

$$=h^{2q+1}/(2q)!\,.\int_{-\frac{1}{2}}^{0}\Psi_{2q}(\theta)f^{(2q+1)}(x+\tfrac{1}{2}h+\theta h)d\theta \quad \ldots\ldots\ldots\ldots(26\text{A})$$

$$=-h^{2q+2}/(2q+1)!\,.\int^{\frac{1}{2}}\Phi_{2q+1}(\theta)f^{(2q+2)}(x+\theta h)d\theta \quad \ldots\ldots\ldots(27)$$

$$=-h^{2q+2}/(2q+1)!\,.\int_{-\frac{1}{2}}^{0}\Psi_{2q+1}(\theta)f^{(2q+2)}(x+\tfrac{1}{2}h+\theta h)d\theta\ldots.(27\text{A})$$

These expressions for Q_q give us various forms for the remainders in (17) and (18).

13. By taking an odd number of the half-strips we obtain formulae for such expressions as $f(a+(n+\frac{1}{2})h)-f(a)$. Generally, if we have such a formula as

$$f(a+nh)=C+h\mu\sigma f'(a+nh)-B_1/2!\,.\,h^2f''(a+nh)+B_2/4!\,.\,h^4f^{iv}(a+nh)$$
$$-\ldots+(-)^qB_q/(2q)!\,.\,h^{2q}f^{(2q)}(a+nh)$$
$$+1/(2q)!\,.\,h^{2q+1}\int_{-\frac{1}{2}}^{\frac{1}{2}}\sigma f^{(2q+1)}(a+nh+\theta h)\,.\,\Psi_{2q}(\theta)d\theta, \quad \ldots\ldots\ldots\ldots(28)$$

where each of the two summations starts from a definite value, and C is a constant, depending on these values, then we have also

$$f(a+(n+\tfrac{1}{2})h)=C+h\sigma f'(a+(n+\tfrac{1}{2})h)+p_1/2!\,.\,h^2f''(a+(n+\tfrac{1}{2})h)$$
$$-p_2/4!\,.\,h^4f^{iv}(a+(n+\tfrac{1}{2})h)+\ldots$$
$$\ldots+(-)^{q-1}p_q/(2q)!\,.\,h^{2q}f^{(2q)}(a+(n+\tfrac{1}{2})h)$$
$$+1/(2q)!\,.\,h^{2q+1}\Big\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\sigma f^{(2q+1)}(a+nh+\theta h)$$
$$+\int_{-\frac{1}{2}}^{0}f^{(2q+1)}(a+(n+\tfrac{1}{2})h+\theta h)\Big\}\Psi_{2q}(\theta)d\theta, \ldots\ldots\ldots\ldots(29)$$

$$f(a+(n-\tfrac{1}{2})h)=C+h\sigma f'(a+(n-\tfrac{1}{2})h)+p_1/2!\,.\,h^2f''(a+(n-\tfrac{1}{2})h)$$
$$-p_2/4!\,.\,h^4f^{iv}(a+(n-\tfrac{1}{2})h)+\ldots$$
$$\ldots+(-)^{q-1}p_q/(2q)!\,.\,h^{2q}f^{(2q)}(a+(n-\tfrac{1}{2})h)$$
$$+1/(2q)!\,.\,h^{2q+1}\Big\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\sigma f^{(2q+1)}(a+(n-1)h+\theta h)$$
$$+\int_{-\frac{1}{2}}^{0}f^{(2q+1)}(a+(n-\tfrac{1}{2})h+\theta h)\Big\}\Psi_{2q}(\theta)d\theta, \ldots\ldots\ldots\ldots(29\text{A})$$

in which, if the summations continue to start from the same values, the value of C is unaltered, account being taken of the additional term in the remainder; and similarly we can pass from (29) or (29A) to (28).

14. We see from (2) and (3) of § 5 that the various remainders depend on the remainders in the expansion of $\frac{1}{2}\coth\frac{1}{2}D$ and $\frac{1}{2}\operatorname{cosech}\frac{1}{2}D$. The following may be taken as specimen formulae:

$$x\coth x=-B_0+2^2B_1x^2/2!-2^4B_2x^4/4!+\ldots+(-)^{q-1}2^{2q}B_qx^{2q}/(2q)!$$
$$-x^{2q}/(2q)!\,.\,x\operatorname{cosech}x\int_0^1 2^{2q}\Psi_{2q}(\tfrac{1}{2}\theta)\cosh x\theta\,d\theta, \quad \ldots\ldots\ldots\ldots(30)$$

$$x\operatorname{cosech}x=P_0-P_1x^2/2!+P_2x^4/4!-\ldots+(-)^qP_qx^{2q}/(2q)!$$
$$-x^{2q}/(2q)!\,.\,x\operatorname{cosech}x\int_0^1 2^{2q}\Phi_{2q}(\tfrac{1}{2}\theta)\cosh x\theta\,d\theta. \ldots\ldots\ldots\ldots(31)$$

The formula in § 12 is the equivalent of the identity

$$0=x-e^x x\operatorname{cosech}x+x\coth x. \quad \ldots\ldots\ldots\ldots(32)$$

15. If $f(x)$ is a function containing only odd powers of x, then by putting $x+\frac{1}{2}h=0$ in (25) we obtain (cf. Whittaker, *Modern Analysis*, § 64)

$$f(x)=xf'(x)-B_1(2x)^2f''(x)/2!+B_2(2x)^4f^{iv}(x)/4!-\ldots$$
$$+(-)^qB_q(2x)^{2q}f^{(2q)}(x)/(2q)!$$
$$+(2x)^{2q+1}/(2q)!\,.\int_0^{\frac{1}{2}}\Psi_{2q}(\theta)f^{(2q+1)}(2x\theta)d\theta, \quad \ldots\ldots\ldots\ldots(33)$$

from which (30) follows by taking $f(x) \equiv \sinh x$; and similarly by putting $x=0$ in (25) and then replacing $\frac{1}{2}h$ by x,

$$f(x)=xf'(0)+p_1(2x)^2f''(x)/2!-p_2(2x)^4f^{iv}(x)/4!+\ldots+(-)^{q-1}p_q(2x)^{2q}f^{(2q)}(x)/(2q)!$$
$$+(2x)^{2q+1}/(2q)!\,.\int_0^{\frac{1}{2}}\Phi_{2q}(\theta)f^{(2q+1)}(2x\theta)d\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots(34)$$

from which (31) follows.

16. Since $p_q=\{1-(\frac{1}{2})^{2q-1}B_q{}^1$, the terms in (18) tend to become equal to the corresponding terms in (17), but of opposite sign. The interval of convergence is therefore the same for both series. Adopting the form of remainder given in (28), we see that, since $|\Psi_{2q}(\theta)|$, in the integral, varies between 0 and B_q, and $B_q/(2q)!$ becomes equal to $2\{1/(2\pi)\}^{2q}$ when q becomes indefinitely great, each series will be convergent if

$$\left(\frac{h}{2\pi}\right)^{2q+1}f^{(2q+1)}(x)$$

becomes indefinitely small when q becomes indefinitely great, for all values of x from $x=a$ to $x=a+nh$, and $f(x)$ and its derived functions are finite and continuous throughout this interval.

If these conditions are not satisfied, it may still be possible to use the series for purposes of approximate calculation. As regards the series in (17), we know * that, if $f^{(2q+1)}(x)$ and $f^{(2q+3)}(x)$ are of constant sign from $x=a$ to $x=a+nh$, the sign being the same in both cases, the remainder after the term in h^{2q-2} is of the same sign as the next term of the series, and numerically less than it; and (by using (x) of § 9) the same property may be proved of the series in (18).

17. The above results may be illustrated by reference to two familiar examples.

(i) Corresponding to

$$\Sigma 1/n=C+\log n+\tfrac{1}{2}n^{-1}-\tfrac{1}{12}n^{-2}+\tfrac{1}{120}n^{-4}+\ldots+(-)^q\frac{B_q}{2q}n^{-2q}+\ldots,$$

where $\Sigma 1/n \equiv 1/1+1/2+1/3+\ldots+1/n$, we have

$$\Sigma 1/n=C+\log\nu+\tfrac{1}{24}\nu^{-2}-\tfrac{7}{960}\nu^{-4}+\ldots+(-)^{q-1}\frac{p_q}{2q}\nu^{-2q}+\ldots.$$

The constant C is the same in the two series. For, if we consider them independently, the same reasoning that gives C as the limit of $\Sigma 1/n-\log n$ in the one case gives it as the limit of $\Sigma 1/n-\log\nu$ in the other; and these two limits are equal. Or, if we compare the two series, the difference between the sums of the terms after the constant in the one case and in the other is

$$(\log n+\tfrac{1}{2}n^{-1}-\log\nu)-\tfrac{1}{2}(B_1n^{-2}+p_1\nu^{-2})+\tfrac{1}{4}(B_2n^{-4}+p_2\nu^{-4})$$
$$-\ldots+(-)^q\frac{1}{2q}(B_qn^{-2q}+p_q\nu^{-2q})+\ldots;$$

and the sum of the terms of this series up to and including the term involving B_q is, by § 12,

$$-\int_{-\frac{1}{2}}^{0}\Psi_{2q}(\theta).(\nu+\theta)^{-(2q+1)}d\theta,$$

which, whatever the value of q, may be made as small as we please by taking ν large enough.

(ii) Similarly in place of

$$\log\Gamma(1+x)=\tfrac{1}{2}\log(2\pi)+(x+\tfrac{1}{2})\log x-x+\tfrac{1}{12}x^{-1}-\tfrac{1}{360}x^{-3}+\ldots$$
$$+(-)^{q-1}\frac{B_q}{2q(2q-1)}x^{-(2q-1)}+\ldots,$$

* Bromwich, p. 325. But, if R is the remainder after a certain term of a series, the error due to the omission of R is $-R$, not $+R$.

we should have

$$\log\Gamma(1+x)=\tfrac{1}{2}\log(2\pi)+\xi\log\xi-\xi-\tfrac{1}{24}\xi^{-1}+\tfrac{7}{2880}\xi^{-3}-\tfrac{31}{40320}\xi^{-5}$$
$$+\ldots+(-)^q\frac{p_q}{2q(2q-1)}\xi^{-(2q-1)}+\ldots,$$

where $\xi\equiv x+\frac{1}{2}$. This may be compared with an expression given by A. R. Forsyth (*Fifty-third Report of British Association* (for 1883), p. 408), which may be written more neatly

$$\log\Gamma(1+x)=\tfrac{1}{2}\log(2\pi)+\tfrac{1}{2}\xi\log(\xi^2-\tfrac{1}{12})-\xi+\tfrac{1}{240}\xi^{-3}-\tfrac{61}{90720}\xi^{-5}+\ldots.$$

W. F. SHEPPARD.

MATHEMATICAL NOTES.

387. [R. 7. b. γ. s. 6.] *Note on* 385 *page* 14.

Half-way down page 15 we read 'Hence before it hits the ground it travels a distance $v\sqrt{\frac{6}{g}}$ feet.' As however the bullet has a velocity of $v\alpha$ downwards this is not accurate.

(If n_1 is about 7 then α is about ·0064 and t the time to reach the ground less than ·2, giving vt about 1·3, so that $n_1<7-\dfrac{12{\cdot}5}{n_1}+1{\cdot}3$, whence $n_1<6{\cdot}4$ instead of 8·4.)

The calculation may be more easily made thus: The time to reach the target is $\frac{x}{v}$ and if the bottom is struck

$$-3=v\alpha\,.\,\frac{x}{v}-\tfrac{1}{2}g\frac{x^2}{v^2}$$

or
$$\frac{16}{2000^2}x^2-\left(\cdot001-4\frac{4{\cdot}5}{x}\right)x-3=0,$$

i.e.
$$x^2-2100x+375000=0;$$

$$\therefore\quad\frac{x}{100}=10{\cdot}5\pm\sqrt{110{\cdot}25-37{\cdot}5},$$

whence
$$x=\frac{10{\cdot}5+\sqrt{72{\cdot}75}}{3}\times100\text{ yds.}=\frac{1900}{3}\text{ yds.}=630\text{ yds. approx.}$$

As however the assumption that the horizontal velocity is constant is very far from reality the calculation cannot give any valuable figures as to the actual range obtainable.

It may be of interest to notice that with the given data, the equation for the height at which the bullet hits the man

$$h=ax-\tfrac{1}{2}g\frac{x^2}{v^2}$$

or
$$x^2-2100\,x+\frac{4{\cdot}5+h}{{\cdot}000004}=0,$$

gives
$$4{\cdot}41>4{\cdot}5+h$$

or
$$h<-{\cdot}09,$$

so that the man is always struck below the middle. C. H. HARDINGHAM.

388. [I. 18. c.] *Powers of numbers whose sum is the same power of some number.*

I wish to add two more early references to the biquadrate identity

$$4^4+6^4+8^4+9^4+14^4=15^4,$$

in order to make its history as nearly complete as possible.

See solution of Quest. 3115, by the proposer, Dr. David S. Heart, *Educational Times*, vol. xxiii. (Nov., 1870), p. 183. This is the earliest publication of it now known to me.

It was also published in the *Reprint*, vol. xiv. (1871), pp. 86-7. These two references were inadvertently omitted in my note on p. 378.

ARTEMAS MARTIN.

389. [L¹. 4.] *The ratio of the lengths of two adjacent tangents to a plane curve.*

1. If O is an ordinary point on a plane curve, M an adjacent point on the curve, T the point of intersection of the tangents at O and M, then $OT/TM \to 1$ when $M \to O$.

2. If O is a point of undulation (or inflexion) at which the tangent has contact of order n with the curve, $OT/TM \to n$.

To prove this, take the tangent and normal at O as axes, and let θ and ψ be the angles made by OM, TM with the x-axis OT. Then

$$OT/TM = \sin(\psi - \theta)/\sin\theta,$$

so that we have to find the limit of $\psi/\theta - 1$. Let $y = f(x)$ be the equation of the curve.

Since y and its first n differential coefficients vanish for $x=0$, and the $n+1$th differential coefficient does not vanish, we have in the neighbourhood of the origin

$$\tan\psi = \frac{dy}{dx} = \frac{x^n}{n!} f^{n+1}(\theta x),$$

$$\tan\theta = \frac{y}{x} = \frac{1}{x} \cdot \frac{x^{n+1}}{n+1!} f^{n+1}(\theta' x)$$

where θ and θ' lie between 0 and 1. Hence $\tan\psi/\tan\theta \to n+1$ when $x \to 0$, and

$$\text{Lt } OT/TM = \text{Lt} \sin(\psi - \theta)/\sin\theta = \text{Lt}(\psi/\theta - 1) = \text{Lt}(\tan\psi/\tan\theta - 1) = n,$$

which proves the proposition.

3. When the cycle of the curve whose origin is O is non-linear, then the limit of OT/TM is *the ratio of the class of the cycle to its degree.* This result includes the two preceding cases.

If q is the degree and p-q the class of the cycle, then if we write

$$x = t^q$$

we shall have $$y = at^p + bt^{p+1} + \ldots\ldots,$$

so that $$\text{Lt } OT/TM = \text{Lt } x\frac{dy}{dx}/y = (p-q)/q.$$

§ 2 is included in § 3, but the proof of case 3 involves the differentiation of a power series, and that for § 2 does not. M. F. EGAN.

390. [L². 1. b.] *Tetrahedral Coordinates.—The conditions necessary in order that the general quadric should be one of revolution.*

Let the quadric be

$$F(\alpha\beta\gamma\delta) \equiv q\alpha^2 + r\beta^2 + s\gamma^2 + t\delta^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha\delta + 2v\beta\delta + 2w\gamma\delta = 0$$

and $ABCD$ the tetrahedron of reference.

$$\lambda F = \left[\alpha\frac{\partial F_0}{\partial\alpha_0} + \beta\frac{\partial F_0}{\partial\beta_0} + \gamma\frac{\partial F_0}{\partial\gamma_0} + \delta\frac{\partial F_0}{\partial\delta_0}\right]^2$$

is the system of quadrics touching $F=0$ round the curve of section by the polar plane of $\alpha_0\beta_0\gamma_0\delta_0$. If the quadric be one of revolution, and $\alpha_0\beta_0\gamma_0\delta_0$ be on the axis of revolution, and under these conditions alone, one of the

quadrics of this system will be a sphere. Then, writing down the conditions for a sphere, we have, if λ_1 is the value of λ for this sphere,

$$\left[\left\{\lambda q-\left(\frac{\partial F_0}{\partial \alpha_0}\right)^2\right\}+\left\{\lambda r-\left(\frac{\partial F_0}{\partial \beta_0}\right)^2\right\}-2\left\{\lambda h-\left(\frac{\partial F_0}{\partial \alpha_0}\right)\left(\frac{\partial F_0}{\partial \beta_0}\right)\right\}\right]/AB^2,$$

and five similar expressions all equal. Put them each $=K$. Then the six equations so formed may be written

$$\lambda(q+r-2h)-AB^2.K=\left[\left(\frac{\partial F_0}{\partial \alpha_0}\right)-\left(\frac{\partial F_0}{\partial \beta_0}\right)\right]^2, \quad \ldots\ldots\ldots\ldots(1)$$

$$\lambda(q+s-2g)-AC^2.K=\left[\left(\frac{\partial F_0}{\partial \alpha_0}\right)-\left(\frac{\partial F_0}{\partial \gamma_0}\right)\right]^2, \quad \ldots\ldots\ldots\ldots(2)$$

$$\lambda(q+t-2u)-AD^2.K=\left[\left(\frac{\partial F_0}{\partial \alpha_0}\right)-\left(\frac{\partial F_0}{\partial \delta_0}\right)\right]^2, \quad \ldots\ldots\ldots\ldots(3)$$

$$\lambda(r+s-2f)-BC^2.K=\left[\left(\frac{\partial F_0}{\partial \beta_0}\right)-\left(\frac{\partial F_0}{\partial \gamma_0}\right)\right]^2, \quad \ldots\ldots\ldots\ldots(4)$$

$$\lambda(r+t-2v)-BD^2.K=\left[\left(\frac{\partial F_0}{\partial \beta_0}\right)-\left(\frac{\partial F_0}{\partial \delta_0}\right)\right]^2, \quad \ldots\ldots\ldots\ldots(5)$$

$$\lambda(s+t-2w)-CD^2.K=\left[\left(\frac{\partial F_0}{\partial \gamma_0}\right)-\left(\frac{\partial F_0}{\partial \delta_0}\right)\right]^2. \quad \ldots\ldots\ldots\ldots(6)$$

From (4), (5), (6) we have

$$\sqrt{\lambda(r+s-2f)-BC^2.K}\pm\sqrt{\lambda(r+t-2v)-BD^2.K}\pm\sqrt{\lambda(s+t-2w)-CD^2.K}=0.$$

Rationalising this we have

$$\begin{aligned}&\lambda^2\{-(r+s-2f)^2-(r+t-2v)^2-(s+t-2w)^2+2(r+t-2v)(s+t-2w)\\&\qquad+2(s+t-2w)(r+s-2f)+2(r+s-2f)(r+t-2v)\}\\&\quad-2\lambda K\{(BD^2+CD^2-BC^2)(r+s-2f)+(CD^2+BC^2-BD^2)(r+t-2v)\\&\qquad+(BC^2+BD^2-CD^2)(s+t-2w)\\&\quad+K^2\{-BC^4-BD^4-CD^4+2BD^2.CD^2+2CD^2.BC^2+2BC^2.BD^2\}=0.\end{aligned}$$

The coefficient of K^2 is $16\Delta_1{}^2$ where Δ_1 is area of face BCD. From equations (2), (3), (6); (1), (3), (5); and (1), (2), (4); we get three similar equations. Eliminating λ, K from each of the four sets of three equations that can be selected from the four equations thus found we have the following conditions necessary for the quadric to be one of revolution.

$$\left\|\begin{array}{l}(r+s-2f)^2+(r+t-2v)^2+(s+t-2w)^2-2(r+t-2v)(s+t-2w)\\\quad-2(s+t-2w)(r+s-2f)-2(r+s-2f)(r+t-2v),\\\qquad BC^2(t+f-v-w)+BD^2(s+v-f-w)+CD^2(r+w-f-v),\ \Delta_1{}^2\\(q+s-2g)^2+(q+t-2u)^2+(s+t-2w)^2-2(q+t-2u)(s+t-2w)\\\quad-2(s+t-2w)(q+s-2g)-2(q+s-2g)(q+t-2u),\\\qquad AC^2(t+g-u-w)+AD^2(s+u-g-w)+CD^2(q+w-g-u),\ \Delta_2{}^2\\(q+r-2h)^2+(q+t-2u)^2+(r+t-2v)^2-2(q+t-2u)(r+t-2v)\\\quad-2(r+t-2v)(q+r-2h)-2(q+r-2h)(q+t-2u),\\\qquad AB^2(t+h-u-v)+AD^2(r+u-h-v)+BD^2(q+v-h-u),\ \Delta_3{}^2\\(q+r-2h)^2+(q+s-2g)^2+(r+s-2f)^2+2(q+s-2g)(r+s-2f)\\\quad-2(r+s-2f)(q+r-2h)-2(q+r-2h)(q+s-2g),\\\qquad AB^2(s+h-f-g)+AC^2(r+g-f-h)+BC^2(q+f-g-h),\ \Delta_4{}^2\end{array}\right\|=0,$$

i.e., the four determinants obtained by taking three of these rows at a time have to vanish. The vanishing of two is sufficient to make the four vanish, so that we have two independent conditions. H. L. TRACHTENBERG.

391. [K. 8. e. : V. 2. d.] *An extension of a theorem in Pappus.*

There is in Pappus (Bk. IV. Prop. 1) a simple and elegant theorem with which I made acquaintance perhaps more than thirty years ago in a German school book. Some years back, happening to mention it at a meeting of distinguished mathematicians, I was surprised to find none of them had heard of it. By this time it may be fairly well known, for it has been pointed out to me that it is in Casey's *Sequel to Euclid*, p. 18. Ex. 20, though without indication of its origin. I also understand that Pappus is little known to British geometers, a defect which may be remedied by the forthcoming edition to be published by Sir Thomas Heath. Recently an extension of the theorem in question has occurred to me which is of interest, I think, as illustrating the principle of algebraic signs in geometry.

The original proposition finds a parallelogram on a side of a triangle equal to the sum of any two parallelograms on the other two sides. Thus it is far more general (*καθολικώτερον πολλῷ*, as Pappus says) than Euclid I. 47, which is a case of it, and the generalisation of that proposition in Euclid VI.

The extension finds a parallelogram on a side of the triangle which is equal to the difference between any two parallelograms on the other two sides.

The original theorem is as follows:

Let $ABDE$ and $ACKH$ be parallelograms described on the sides AB and AC of the triangle ABC. Let DE and KH, produced if necessary, meet in L. Join AL. From B and C draw BO and CM parallel to AL, meeting DE and KH, or these lines produced, in O and M respectively. Join OM. Then $BCMO$ is the required parallelogram. If LA is produced to meet AB in P, the proof easily follows from Euclid I. 35.

To generalise the theorem we may consider the three parallelograms as three figures forming a system related by the given construction, and seek for generality by varying the position of one of them, while we keep the same connecting construction. If now we move the parallelogram on the base so that one of the angles comes within the corresponding angle of the triangle, P will move to the outside of AB, and the parallelogram on the corresponding side will come from the outside of the triangle and fall partly within it; and we shall now find that the parallelogram on the base is equal to the difference of the parallelograms on the sides. We thus get a construction for the parallelogram on the base equal to the difference between the given ones on the sides.

Further, if we consider the areas of parallelograms described on opposite sides of the same line as of opposite sign, the difference between the parallelograms in the second theorem would be their algebraic sum. Also, in the first figure the proper enunciation would be that the algebraic sum of the parallelograms so described is zero, and this obviously would also be the enunciation of the second theorem, and so the two are embraced in one formula.

It may of course be easily verified that this all holds good if we start with the parallelogram on the base described outside the triangle.

J. Cook Wilson.

392. [K. 2. e.] *A Nine-point Circle.*

Consider any four lines 1, 2, 3, 4.

The circumcentres O_1, O_2, O_3, O_4 of the four triangles formed by them lie on a circle—call it the centre circle.

The circumcircles of these triangles have a common point—call it M—on the centre circle.

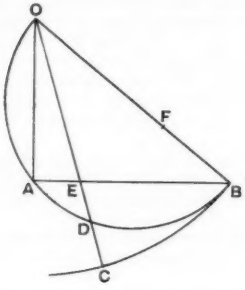
The circumcircle of the triangle formed by 1, 2, 3 cuts the centre circle again in a point N_1, which is the pole of the Simson Line parallel to 4.

We thus have *nine* concyclic points, O_1, O_2, O_3, O_4, M, N_1, N_2, N_3, N_4.

A remarkable circle. Has any geometer tackled it? W. Gallatly.

393. [K. 21. b.] *Mechanical Construction for the Trisection of an Angle.*

Let AOB be the angle. Describe a semicircle on OB. Join AB. With O as centre, draw the arc BC. Through O draw OC in such a way that the part intercepted between BA and arc BC is bisected by the arc BDA. (This can be effected most accurately by means of a flat rule.)



Then OC is the required trisector.

$$OC.OE=OE^2+OE.EC,$$

or $$OB.OE=OE^2+2OE.DE$$

$$=OE^2+2AE.EB=OA^2+AE^2+2AE.EB$$

$$=OA^2+AB^2-EB^2=OB^2-EB^2,$$

$$i.e.\ OB(OB-OE)=EB^2.$$

Make $OF=OE$, and we have

$$OB.BF=BE^2.$$

The circle round OEF will touch AB at E.

Hence $$\hat{OEA}=\hat{OFE}=\hat{OEF}.$$

The supplement of twice $\hat{OEF}$ (which is $\hat{EOF}$) is therefore twice the complement of $\hat{AEO}$ (which is $\hat{AOE}$); hence $\hat{EOF}$ is twice $\hat{AOE}$; that is, OC is a trisector.

I have not seen this mechanical construction anywhere in print. It appears to be simple as compared with the constructions I have come across, and I publish it after a hesitation of some years.

Christ's College, Cambridge. INAYAT ULLAHKHÁN.

394. [K. 13. a.] The following method of finding the length and equation and the coordinates of the extremities of the shortest distance between two lines whose equations are given is perhaps as short as any for numerical examples, although it leads to heavy work in the general case.

Let the given lines be $$\frac{x-1}{1}=\frac{y-9}{-2}=\frac{z-5}{1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$\frac{x-6}{7}=\frac{y+7}{-6}=\frac{z}{1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The coordinates of any two points on (1) and (2) respectively may be written $$(1+r,\quad 9-2r,\quad 5+r)$$

and $$(6+7s,\ -7-6s,\ s).$$

Hence the direction-cosines of the line joining them are proportional to

$$r-7s-5,\quad -2r+6s+16,\quad r-s+5.$$

If this line is the shortest distance, the conditions of perpendicularity give

$$(r-7s-5)-2(-2r+6s+16)+(r-s+5)=0$$

and $$7(r-7s-5)-6(-2r+6s+16)+(r-s+5)=0.$$

Solving these, we find $r=2$, $s=-1$, so the extremities of the shortest distance are $(3, 5, 7)$ and $(-1, -1, -1)$.

Hence the length of the shortest distance is $\sqrt{116}$, and its equation is

$$\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}.$$

H. PIAGGIO.

395. [**V; R. 4.**] (*Re* Note 344, p. 151, vol. vi.) The criticism of the questions was put forward after discussion with three other teachers. If the simple solution stated by Mr. Heawood be allowed in this case, it cannot be forbidden for many other questions, which become then mere verifications and are brought down to Stage I. standard. The responsible examiner was appealed to, to settle the point, but he declined to answer, stating that he was debarred *ex-officio*. The solution is certainly too simple for Stage II.; compare it with the second question in vol. v. p. 164. M. E. J. GHEURY.

396. [**M¹. 3. i.** α.] The following results with regard to the area of "borders" may be of interest.

(1) If two rectilinear polygons have their sides parallel each to each and a constant distance apart, the border so formed is made up of trapeziums. Hence, for such borders

$$A = h \times P,$$

where A is the area, h the width and P the mean of the outer and inner perimeters of the border.

(2) If two closed curves have continuous curvature and a common evolute, so that the normal distance between them is constant, let (x, y) be a point on the inner. Then to the first order of small quantities the contiguous point is

$$(x+\delta x,\ y+y_1\delta x),$$

and the corresponding points on the outer are

$$(x-h\sin\psi,\ y+h\cos\psi)$$

and $(x-h\sin\psi+\delta x-h\cos\psi\,.\,\psi_1\delta x,\ y+h\cos\psi+y_1\delta x-h\sin\psi\,.\,\psi_1\delta x)$.

If these points be referred to the tangent and normal at (x, y), they become

$$(0, 0)(\delta x\sec\psi, 0)(0, h) \quad \text{and} \quad (\delta x\sec\psi - h\psi_1\delta x, h),$$

which form a trapezium.

Hence, for such a border also, $A = h \times P$.

(3) For no other classes of borders does the qualification "of constant width" appear to have any obvious meaning, though the theorem might be extended to curvilinear polygons with certain restrictions.

(4) The excess of the outer over the inner perimeter for all borders of class (2) is $2\pi h$, a result which is of practical interest in the case of running-tracks.

Haileybury. C. W. ADAMS.

397. [**I. 17.**] Vol. vi. No. 99, July 1912, p. 338, notes 373, 381.

Identity (v) should read

$$(x^6+8x^4y^2)^2+(36x^2y^4+162y^6)^2$$
$$=(x^6+4x^5y+16x^4y^2+42x^3y^3+72x^2y^4+108xy^5+162y^6)$$
$$\times(x^6-4x^5y+16x^4y^2-42x^3y^3+72x^2y^4-108xy^5+162y^6).$$

It may be noticed that the left side of identity (i) is not only a sum of *two* but also a sum of *four* squares, viz. $(x^6)^2+(4x^2y^4)^2+(4xy^5)^2+(2y^6)^2$; and the left side of (v), as corrected, is a sum of *six* squares, viz.

$$(x^6)^2+(4x^5y)^2+(8x^4y^2)^2+(36x^2y^4)^2+(108xy^5)^2+(162y^6)^2.$$ G. OSBORN.

398. [**K. 7. d**; **M¹. e.** δ.] I was much interested in the proof given by Prof. A. C. Dixon and in the correlative theorem given by the Rev. John J. Milne of the theorem published by me in the *Mathematical Gazette*, Jan. 1911, p. 386. I have now obtained a still more elementary proof of the theorem above referred to, and the method applies equally well to the corresponding theorem in three dimensions, and more generally in space of n dimensions.

Let S and S' be two conics or two quadrics. Then a polar of $S+\lambda S'=0$ is plainly of the form $P+\lambda P'=0$, where P and P' are linear in the coordinates. *Hence the polars of any two points with respect to members of the system* $S+\lambda S'$ *form two homographic systems*, since each polar member is defined linearly by the same parameter λ. .....(1)

Theorem in space of two dimensions.

If P, Q, R, S be four points and if O_1 and O_2 be conjugate points with respect to each member of the system of four-point conics $PQRS$, then

$$O_1[PQRS]=O_2[O_1ABC],$$

where A, B, C are the vertices of the self-conjugate triangle.

Consider the four conics of the above system, viz.: $S_1 \equiv$ the lines APQ, ARS; $S_2 \equiv BPR$, BSQ; $S_3 \equiv CPS$, CQR; $S_4 \equiv O_1PQRS$.

Then the polar lines of O_1 with regard to S_1, S_2, S_3, S_4 are respectively O_2A, O_2B, O_2C, O_2O_1. .....(2)

Also the polar lines of P with regard to S_1, S_2, S_3, S_4 are respectively PQ, PR, PS, tangent at P to conic O_1PQRS. .....(3)

Hence, by (1), (2), (3),

$$O_2[ABCO_1]=P[QRSP]$$
$$=O_1[QRSP] \quad \text{(by cross-ratio property of conics).}$$

Theorem in space of three dimensions.

Let the two quadrics S and S' have as their intersection the quadriquadric SS', and let $ABCD$ be the common self-conjugate tetrahedron.

Let P be any given point in space having as the line of intersection of all its polar planes with respect to members of the system $S+\lambda S'$ the line p. Let Q be any point on the quadriquadric and let q in the same way be the line common to all its polar planes.

Consider the four cones of the system $S+\lambda S'$ Then the polar planes of P and Q with regard to these four cones are pA, pB, pC, pD and qA, qB, qC, qD.

Hence, by (1), $\qquad p[ABCD]=q[ABCD]$.

But qA, qB, qC, qD are evidently the four planes drawn through the tangent q to the quadriquadric to touch the curve again.

Hence the cross-ratio of the four tangent planes that can be drawn from any tangent to a quadriquadric to touch the curve again is constant.

Also, since $p[ABCD]$ is constant, we see that *all the lines p belong to the Tetrahedral Complex defined by the tetrahedron $ABCD$.*

WILLIAM P. MILNE.

399. [K. 1. γ.] *Given* $\triangle BAC$; BC fixed in position; angle A of constant size; BA produced to P so that $AP=AC$.

To find the locus of P.

It is a circular arc such that $B\hat{P}C=\frac{1}{2}A$.

Now consider the limiting positions:

(1) When A coincides with C, P also coincides with C; $\therefore$ C is on the locus.

(2) When A coincides with B, AB becomes a tangent at B to the $\odot BAC$, viz. BT, and T is a pt. on the locus if $BT=BC$.

The meaning of the arc TBC of the locus-circle remains to be interpreted.

In the case of the orthocentre of $\triangle ABC$, the complete locus is obtained by including the locus of the orthocentre of $\triangle BDC$ in the segment supplementary to ABC.

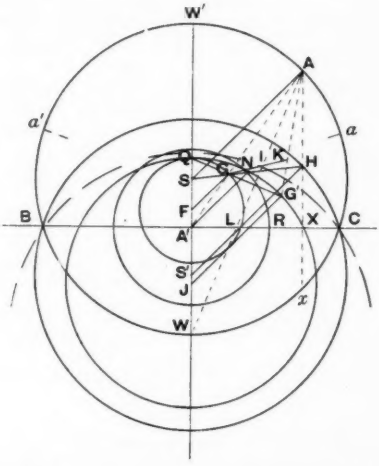
In the case of the incentre, the complete locus-circle is obtained by including the excentre.

In this case, if D is a pt. moving on the arc supplementary to ABC, and if DB is produced to P so that $DP=DC$, then the resulting positions of P complete the locus-circle.

F. C. BOON.

400. [K. 1. b. γ.] Note 362, page 221, vol. vi.

State the problem thus: Given the fixed base BC and the locus of the vertex A a fixed circle BCA, required the locus of the orthocentre H. Let the radius of the circle $BCA = R$ and let WW' be the diameter bisecting BC at right angles. From S the centre set off in the direction SA', $SS' = 2SA'$. With S' as centre, radius R, draw circle BHC; then $S'H$ parallel to SA gives H the orthocentre. The circumference of S' is the locus of H.



Let α and α' be the ends of diameters from B and C; then in the arcs $\alpha' W' \alpha$ and CWB the angle A is the supplement of the angle BHC, and in the arcs αC and $\alpha' B$ the angles are equal.

The nine-point circle centre locus is a circle, centre A', radius $= \frac{1}{2}R$.

The centroid locus is a circle whose centre F is $\frac{1}{3}A'S$ from A', radius $= \frac{1}{3}R$.

The I and I_1 locus is a circle, centre W, radius $WB = WC$.

Let G' be the centroid of the triangle B_1C_1A, where B_1C_1 is antiparallel to BC and $= BC$.

The locus of G' is a circle, centre J, where $SJ = \frac{2}{3}R$ and its radius $= WF = R - \frac{2}{3}SA'$. Since G and G' coincide at Q, $QJ = R - \frac{2}{3}SA'$.

$G'A$ is a symmedian of ABC.

GI passes through K, the symmedian point, for

$$\Sigma(AG) : \Sigma(GA') = \Sigma(AI) : \Sigma(IL) = \Sigma(AK) : \Sigma(KR) = 2 : 1.$$

Hence the four lines AA', QG', SH and IK are concurrent at G.

SA, $S'H$, FG, $A'N$ and JG' are all parallel.

The two extreme cases, BC a diameter and BC a tangent, are interesting.

In drawing a large figure for these loci and others which accompany them, take R and $A'S$ multiples of 3.

The locus of K is an ellipse whose axes are easily got with the aid of the fixed tangents at B or C, and the moving tangent at A. W. F.

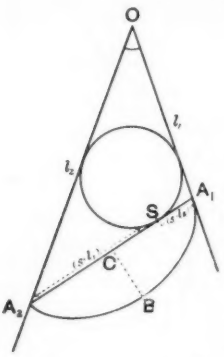
401. [K. 15. b.] Given a right circular cone (see figure), to find the volume V of a slant cone, vertex O, and elliptic base A_1BA_2; and to prove

that it is a Geometric Mean* between the volumes of the *right* cones C_1, vertex O, slant height OA_1 and C_2, vertex O, slant height OA_2.

Let the focal sphere touch A_1A_2 at S;

then, if $OA_1=l_1$ and $OA_2=l_2$,

$A_1S=s-l_2$ and $A_2S=s-l_1$,



where s=the semi-perimeter of the triangle OA_1A_2.

Hence $A_1S.A_2S=(s-l_2)(s-l_1)$.

$\therefore\ CB^2=(s-l_2)(s-l_1)$.

Also $\sin^2\alpha=\dfrac{(s-l_2)(s-l_1)}{l_1l_2}$,

where $A_1\hat{O}A_2=2\alpha$.

$\therefore\ CB^2=l_1l_2\sin^2\alpha$. .....(1)

Let p be the length of the $\perp^r$ from O to A_1A_2;

then $p\,.\,A_1A_2=l_1l_2\sin 2\alpha$. .....(2)

Now $V=\dfrac{1}{3}\pi\,.\,CB\,.\,CA_1\,.\,p$

$=\dfrac{\pi}{6}\,.\,\sqrt{l_1l_2}\sin\alpha\,.\,l_1l_2\sin 2\alpha$,

from (1) and (2).

Also $C_1=\dfrac{\pi}{3}\,.\,l_1{}^2\sin^2\alpha\,.\,l_1\cos\alpha=\dfrac{\pi}{6}\,.\,l_1{}^3\sin 2\alpha\,.\,\sin\alpha$,

C_2 $=\dfrac{\pi}{6}\,.\,l_2{}^3\sin 2\alpha\,.\,\sin\alpha$.

Hence $V^2=C_1C_2$.

Corollary. It is clear that the theorem is equally true for an elliptic cone, as can be seen at once by imagining the cone to be contracted or expanded in the direction $\perp^r$ to the plane A_1OA_2, so that its right sections become ellipses.

Thus, if k is the coefficient of contraction or expansion,

CB becomes $k\,.\,CB$.

Hence V ,, $k\,.\,V$.

Similarly for C_1 and C_2;

whence $V^2=C_1C_2$ as before.

H. CRABTREE.

Charterhouse.

402. [R. 1. d.] *A Stern Chase.*

A cruiser A, speed V knots, sights a cruiser B (of inferior speed v knots) and gives chase. B makes straight for a friendly port, on a given course. Find the course on which A will close within a given range a in the minimum time.

This problem, a slight extension of one contributed by Mr. R. M. Milne to *A First Dynamics*, is not readily amenable to analytical treatment, and I confess to a hope that some readers of the *Gazette* will share the pleasure which I experienced in obtaining a simple geometrical solution of a "practical" problem.

Set off AC to represent a velocity equal and opposite to that of B.

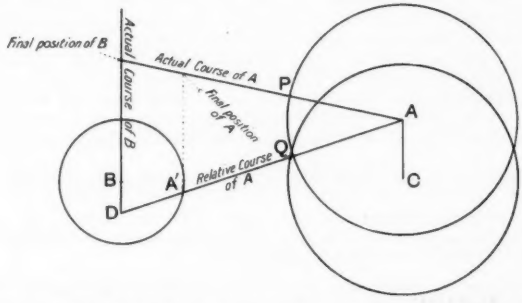
Draw a circle, centre A, radius to represent V.

*So far as the writer is aware this result was first arrived at a few years ago by Mr. Alfred Lodge, of Charterhouse, but by a different method; it is now published with his permission.

Then, if P is any point on this circle, and PQ is drawn parallel and equal to AC, we see that AQ represents the velocity and course of A relative to B, and that the locus of Q is a circle, centre C, radius representing V. Draw a circle, centre B, radius a.

If any line AQA' cuts this latter circle in A', then AA' represents (on the scale for distances) the distance A has to run on the relative course before closing and, AQ representing the velocity, it follows that the time taken to close varies as $\frac{AA'}{AQ}$.

For a minimum value of this ratio two consecutive lines AQA', $AQ'A''$ give the same value to the ratio, so $\frac{AA''}{AQ'}=\frac{AA'}{AQ}$ and $A'A''$ is parallel to QQ'; that is, the tangents at A' and Q must be parallel. In other words, near the critical point the distance and the speed are altering at the same rates.



Thus the problem depends on the geometrical problem: "Through a given point A to draw a straight line cutting two given circles in Q and A', so that the tangents at Q and A' may be parallel."

Set off BD parallel to AC and equal to $\frac{v}{V}a$.

The triangles BDA' and CAQ are similar, hence the radii BA', CQ are parallel, and hence the tangents at Q and A' are parallel.

The geometrical problem has obviously four solutions. C. S. J.

403. [V. 1. a.] *How to approach Euclid I.* 32.

I. Four methods of dealing with this important proposition, or rather with part of it, the property that the angle sum of a triangle is two right angles, are in current use.

1. Measurement. The proposition is inferred, as Mr. Newbold says, from the fact that the mean of three or four measurements is 179°.

2. Rotation. A person walking round the triangle has made a complete turn.

3. Deduction from an intuitive property of parallels—parallel straight lines make equal angles with any straight line crossing them.

4. Euclidean deduction from his axiom, or an equivalent one of minimum type.

Experiment and trial play an important part in mathematical investigation, but the first method, pure and simple, must be pronounced degrading. It is not merely illogical, but fraudulent. The implication is that a boy discovers the theorem from his measurements. In fact he makes a loose verification of a teacher's statement.

If this statement appears harsh, consider the matter further. The theorem is the basis of all the surveyor's methods of verification. To a boy these methods are typified by a test of the accuracy of his protractor, and of his powers of using it. Now we had good leading in the laugh with which we received Professor Turner's yarn of the Egyptian clock which was set by the watch which was set by the gun which was fired by the clock. May we laugh at the teacher whose pupils test their protractors by a theorem obtained by measurement with the same protractor?

Most grown-up people find the rotation method convincing and clear. Moreover, there is no doubt that if we were starting elementary mathematics with a clean slate we should for other reasons define *the* angle of a triangle or polygon, plane or spherical, as what is now called the exterior angle, and thereby increase the directness of the rotation argument. That the method is gravely lacking in logical soundness troubles me but little.

My impression, however, gained, I admit, from limited experience, chiefly with some of my own children, is that children do not find the argument either convincing or clear. It does not make the same appeal to them that it makes to adults. This is of course a question of fact.

The deduction from intuitive properties of parallels is clear and short, and my present opinion is that this is the right way to establish the proposition.

The argument against endeavouring to establish geometry for beginners by any method resting on a minimum of postulates has been stated very clearly by Mr. Carson in recent issues of the *Gazette* and need not be further discussed.

In my opinion a conclusion and a definite pedagogic problem emerge clearly.

Two methods of showing that the angles of a triangle make up two right angles deserve consideration. Which of these two makes the stronger appeal to beginners is worthy of careful investigation.

II. Indulging in speculation, the conjecture is fairly obvious that the sequence of discovery was

(1) The angles of a rectangle make four right angles;

(2) The angles of any right-angled triangle, the half of a rectangle, make two right angles;

(3) Any triangle may be made from two right-angled triangles, and its angle sum is that of the two right-angled triangles less the two right angles.

The impossibility of *proving* (1) might be pointed out at a fairly early stage.

Nor should I hesitate to try, at a fairly early stage, to convince pupils that the smaller the triangle the more nearly must the final proposition hold.

For let the sum of the angles of a triangle be S. Divide it into two by any line through a vertex. The average angle sum is $\frac{1}{2}(S+\pi)$.

Repeating the process we find that the average angle sum of 2^n triangles making up the original triangle is

$$\pi+\frac{S-\pi}{2^n},$$

and the second term tends towards zero as n increases.

III. The proposition may not be true.

It is nearly true in a sense, but there is much virtue in "nearly." We think of $\frac{1}{10^6}$ as small. But consider m, denoting the product of all the integers from 1 to a million. We call m large, m^m larger, m^{m^m} larger still, but these are only the first three of a series of numbers, and, denoting them by $m_1, m_2, m_3, \ldots$ we can write down the symbol m_m. This really begins to look like a large number. Nor is it a *mere* abstraction. Several of its properties are obvious. It is not divisible by 5555567 (by any prime greater

than a million). Now suppose the true formula for the angle sum of a triangle is

$$\pi+\frac{A}{m_m},$$

no experiment could ever detect the falsity of I. 32.

Of course the question of jesting Pilate comes in at the last.

All that we can say is: I. 32 fits all the known facts, and is simple. A formula such as

$$\text{Angle sum}=\pi+\frac{A}{m_m}$$

fits all the known facts, but is less simple.

A consequence of the Euclidean doctrine of parallels—that of the unlimited existence of similar figures involves difficulties. The existence of similar figures involves the absence of an absolute unit of length. Adopting the corresponding doctrine of the non-existence of an absolute unit of time, we see that, in a stereoscopic cinematograph showing the earth's precessional period of 26,000 years in 5 seconds, events would move more quickly, her history would be the same, and a step further would realise M. Anatole France's conception of the stellar universe in a nutshell with the same amount of wretchedness that it contains now. Thus the Euclidean doctrine may after all not always seem to be the simplest.

ANSWER TO QUERY.

[83. Vol. vi. p. 379.] I. Let U_r denote the number of arrangements of 4 Kings and 4 Queens, in which there are r contacts between a King and a Queen.

Then
$$U_{2r}=2\,.\,{}_3C_r\,.\,{}_3C_{r-1}(4!)^2,$$
$$U_{2r+1}=2\,.\,{}_3C_r{}^2(4!)^2.$$

From any one arrangement with r contacts, we can, by means of the 44 remaining cards, form V_r arrangements in which each contact will be broken, where

$$V_r=\{{}_{52}P_8-{}_rC_1\,.\,{}_{51}P_7+{}_rC_2\,.\,{}_{50}P_6-\ldots+(-1)^r{}_{52-r}P_{8-r}\}.$$

$\therefore$ chance of an unfavourable case is

$$\sum_{r=1}^{7}U_rV_r/52!,$$

which works out to 300,684,703/585,307,450.

$\therefore$ required chance is $\dfrac{284,622,747}{585,307,450}=\cdot486\ldots.$

II. With n cards in addition to the 4 Kings and 4 Queens, let $f(r, n)$ denote the number of arrangements in which there are r contacts between a King and a Queen.

Then $f(r, n+1)=(n-r+9)f(r, n)+f(r+1, n).\quad r=0,\ 1,\ 2,\ldots 7,$

with $f(0, 0)=0$; $f(r, 0)=U_r$, where U_r has the same meaning as in (I).

These equations lead to

$$\sum_{r=1}^{r=7}f(r, n)=(4!)^2\{\tfrac{1}{1}\tfrac{1}{8}(n+7)!-\tfrac{2}{3}(n+6)!+\tfrac{2}{6}\tfrac{3}{6}(n+5)!-\tfrac{6}{6}\tfrac{7}{6}(n+4)!$$
$$+16(n+3)!-10(n+2)!+2(n+1)!\},$$

and chance of a favourable case is

$$\sum_{r=1}^{r=7}U_{r,44}/52!,$$

which is 284,622,747/585,307,450. G. N. Bates.

REVIEWS.

History of the Theories of Aether and Electricity. By Professor WHITTAKER. 12s. 6d. net. 1910. (Longmans.)

In one of Zola's works (I think in *L'Oeuvre*) the artist who is bravely working on, indifferent to the adverse verdict of the present, and confident that the future will do him justice, is suddenly assailed by a terrible misgiving, What if posterity should be as stupid as his contemporaries? Would there be any inspiring motive left?

It is to be feared that in the matter of scientific reputations, posterity is sometimes as lacking in appreciation as the public of the time. Few of us probably knew the name of John Michell until we read in Professor Whittaker's book that he was the discoverer of the law of force between magnetic poles, that he devised the "Cavendish" experiment, invented the torsion balance, and taught William Herschel how to grind mirrors. He is in the *D.N.B.* all right, but I cannot find him in the recently-published *Collected Works of William Herschel*, which has a very careful biography; his relations with the great astronomer seem to have escaped Dr. Dreyer's vigilance. We have been almost as much to blame as Michell's "collegiate contemporaries and successors, who silently acquiesced when his discoveries were attributed to others, and allowed his name to perish entirely from Cambridge tradition" (p. 167-8). And yet Michell was the "only natural philosopher of distinction who lived and taught in Cambridge" during the whole century between the death of Newton and the scientific activity of Green. One would have thought that the dark background itself was sufficient to throw him into relief; but the obscurity seems to have swallowed him also.

If such injustice can be repaired, Professor Whittaker has done his utmost towards reparation. His book is probably the greatest act of piety towards the past which has been produced in this generation. There are nearly 500 pages packed full of information; and it is clear that a paragraph or even a sentence often represents a big memoir. He tells us not only of men almost forgotten like Michell, but new facts about men whose names we know. Not only did Michell anticipate Coulomb by discovering the law of magnetic attraction, but Priestley also anticipated him with that of electrostatic attraction (pp. 50-51). Cauchy, who is generally regarded almost solely as a pure mathematician, made some brilliant discoveries in the elastic-solid theory of light, including the labile or contractile aether which was long afterwards rediscovered by Lord Kelvin (pp. 158-60). Coming to more modern times, the author has insisted on the claims of Fitzgerald, who was the real pioneer in much modern physics that goes by the names of others. On the other hand, Englishmen have as yet done scant justice to the Continental work on electrons which was started by Weber in 1846 (p. 228).

The book must have been an enormous piece of work. We can only be grateful that the distinguished author got the opportunity to carry it through in the seclusion of Dunsink before he returned to the busy world in Edinburgh. Shall we be able, with his help, to maintain a juster appreciation of our predecessors? It would seem advisable to keep the book on one of the easily accessible shelves of the study, where it may be referred to constantly. Even then the stream of new ideas may hurry us on.

"In order to represent natural phenomena without introducing this contingent element, it would be necessary to abandon the customary three-dimensional system of co-ordinates, and to operate in four dimensions. Analysis of this kind has been devised, and has been applied to the theory of the aether; but its development belongs to the twentieth century, and consequently falls outside the scope of the present work" (p. 448).

Goodbye! then, to the Victorian era, and its commonplace three dimensions. The poets, we learn, have already closed its volumes of poems and opened a new account with "Georgian" poetry. So also on the scientific side our king is to have four dimensions—perhaps even an appropriate fifth? If the pace gets too hot for us, Professor Whittaker's book may at any rate provide a wholesome sedative.

H. H. T.

Elementary Mechanics. By G. Goodwill, B.Sc. With Preface by Professor F. S. Carey. Pp. 230. 4s. 6d. 1912. (Clarendon Press.)

Certain portions of this book and especially the preliminary chapters have a charm and freshness which cannot fail to attract the teacher who wishes his pupils to get hold of the fundamental notions of the subject. The author seems to be a convinced believer that a clear conception of the law for compounding vectors and the quantities that obey the vector law is the key to the world of mechanics; consequently a great deal of the book will be appreciated only by the boy who has some real mathematical ability.

Passing over the first chapter, which is devoted to the compounding of velocities, we come to the real introduction to mechanics in Chapter II. The novelty of the method may be indicated by quoting the three principles or axioms on which the treatment is based: (i.) "Change of velocity of two bodies is due to action upon it by another body; (ii.) The changes of velocity of two bodies produced by their mutual action are in opposite directions and in a definite ratio; (iii.) This definite ratio is the ratio of what are called the masses of the bodies, by which the changes are produced." The author thus gets at mass, so to speak, through the medium of momentum, and the apparatus which he uses for his momentum experiments is admirably adapted for his purpose. The vector balance, as he calls it, is a sort of glorified ballistic pendulum, and consists simply of two or more spheres attached to long vertical cords, so that they are just in contact when the spheres hang vertically. By the aid of this apparatus the truth of the vector addition of momenta may be as readily illustrated as the vector addition of forces by the aid of some weights, pulleys and a piece of string.

But, when all is said and done, it will be very generally admitted that though vector notation is admirable for stating and generalising results, yet for purposes of investigation and, indeed, for the mere working of numerical examples the very generality of the method seems to make its application awkward. Many of the examples in this text-book bear this out.

On the practical side, the book is as deficient as it is strong on the theoretical side. The three or four pages devoted to machines give one the impression that the author (like the Sage of Syracuse) imagines that he is degrading his subject when he applies it to any useful purpose. On page 100 are illustrated three so-called systems of pulleys, which the author states are "the most common arrangements." In the next edition the author would be well advised to substitute "least" for "most." We doubt if any of these arrangements have ever been seen outside an examination-room. There is no adequate treatment of friction, and little or no reference to the important part it plays in machines. In this connection there is a very misleading statement on p. 68 line 4, where an assumption that is only approximately true is tacitly made, and an equation that is really a vector equation is written as a scalar one.

A Course of Elementary Practical Physics. By H. V. S. Shorter. Part I. **Mensuration, Mechanics, and Hydrostatics.** Pp. 111. Price 2s. Part II. **Heating and Light.** Pp. 216. Price 3s. (Oxford Clarendon Press.)

These two volumes are in the form of laboratory notes for the performance of a definite series of experiments on the part of the boy; to each experiment questions are added to show that he understands what he has been doing. The experiments are carefully graduated and well thought out, but many of them could not be performed without more definite instruction than that supplied in the text, unless the boy had already seen a similar experiment performed in class. We do not doubt that the author has obtained good results from this course, but at the same time we cannot help thinking that in another school, with different apparatus and different methods of instruction, the value of these books would be greatly discounted. Some of the experiments would be much more intelligible if a diagram were supplied.

Syllabus of Mathematics accepted by the Society for the Promotion of Engineering Education at the Nineteenth Annual Meeting held at Pittsburg, Pa., June, 1911. Pp. 136. (Office of the Secretary, Ithaca, N.Y.)

The object of this work, as enunciated in the preface, is to give "a synopsis of those fundamental principles and methods of mathematics which in the opinion of the committee should constitute the minimum mathematical equipment of the

student of engineering." The book contains in expanded form the results and processes that may be found in such a well-known publication as Molesworth's *Pocket-Book of Engineering Formulae*: hints as to the method of proof are given occasionally, and in some cases the full proof is supplied. There is some misleading work on p. 84 in the discussion of the geometrical meaning to be given to $\frac{dy}{dx}$ in the case of the curve $y=f(x)$. There is no reference to graphical work for solving equations or for the determination of the law connecting the two interdependent quantities, and generally the work seems to show more evidence of the hand of the professional mathematician than of the teacher of engineering.

Columns and Struts. By W. ALEXANDER, M.Inst.C.E. Pp. xii+267. 10s. 6d. net. (E. and F. Spon.)

Practically all existing formulae for determining the working stress in a column or strut are based on Euler's Theory for long columns. This theory, as is well known, gives only the smallest force which will just hold the strut bent, when once curvature has been produced, and it leads to the conclusion that the same force will keep the strut bent whether the curvature be great or small. Euler's Theory takes no account of the form of the cross-section of the strut, and the above-mentioned deduction from it, namely, that a strut passes into a state of neutral equilibrium after bending has once commenced, is certainly not true generally, though it may be approximately true for very long solid struts with simple forms of cross-section.

In this work the author carries out investigations connecting the load and the deflection of columns, basing his equations on the maximum permissible fibre stress of which the material is capable. He pays special attention to the form of the cross-section, and discusses in detail the efficiency of most of the commoner sections.

With regard to the theoretical part of the work in which readers of the *Mathematical Gazette* will be chiefly interested, the author cannot be congratulated on the notation which he has adopted. The needless multiplicity of suffixes and dashes is most confusing, such symbols as p''_δ and $I_{0\delta}$ recurring with irritating frequency.

Here and there symbols are carelessly introduced without any hint as to their meaning: thus, on page 12, 'M' is used for Bending Moment, and on page 15 'r' is used for radius of curvature, but the reader is left to supply the interpretations for himself. Many of the figures in the text are too small, and the lettering of the diagrams is in nearly all cases unsatisfactory, owing to the large size of the letters employed.

Readers of this work will turn with interest to Chapter XII., in which is given an account of the causes that led to the failure of the strut in the Cantilever Bridge over the St. Lawrence at Quebec, which made the structure collapse on Aug. 29th, 1907, while in course of erection. The reason of the failure of this member was fully discussed in the Report of the Royal Commission appointed to investigate the cause of the disaster, but the author gives an independent investigation into the question. It was known that this particular strut, which was 57 feet long, was showing signs of weakness: in July it had buckled sideways about ·5 of an inch, in August ·75 of an inch, and two days before the accident occurred the buckling amounted to 2·25 inches. Working from these data and the known dimensions and scantlings, the author shows that failure was due to the omission to provide for the stress in the bracing bars of the strut and their connections. The conclusion arrived at from the theoretical investigation is substantiated by an examination of the drawings and photographs of the parts of the wreckage.

The subject of compression stress in bars is difficult, owing to causes that are only partially known. Any inequality on the two sides of the supports of a strut causes that side which has the greater stress to yield more than the other, so that the resultant thrust tends to get more and more eccentric. The conditions of the fixed supports, the actual working loads, and the wind pressure are additional factors of uncertainty.

The author has done good service in throwing aside the old semi-empirical formulae, which have done duty since Rankine's day, and the book should be of great service to any engineer who has had sufficient theoretical training to follow the somewhat abstruse analyses. R. M. M.

L'étoile magique à 8 branches et les étoiles hypermagiques impaires. By C. SALOMON. Pp. 24. 1 fr. 50 c. 1912. (Gauthier-Villars.)

These arithmetical curiosities show a pretty extension of the idea of a magic square. A device is given for their construction by superposing two simpler figures. Half the pamphlet is taken up with diagrams.

Die Lehre vom Flächeninhalt in der allgemeinen Geometrie. By A. FINZEL. Pp. 38. M. 1.20. 1912. (Teubner.)

This pamphlet develops the theory of area on the basis of Schur's system of plane postulates, without using the idea of continuity or referring to the doctrine of parallels.

Euclid's second and third axioms are replaced by two definitions of equality by addition and equality by subtraction, depending on dissections into congruent polygons; the second of these makes it possible to dispense with the axiom of Archimedes. Area is defined in terms of the sum of the angles of a triangle in non-Euclidean space, and in terms of the base and altitude in Euclidean space; and it is formally proved that polygons with the same area are equal according to the definitions. In the last chapter the results are verified in the notation of the integral calculus. H. P. H.

Elementary Trigonometry. By Dr. R. S. HEATH. Pp. 219. With or without Answers. 3s. 6d. 1912. (Clarendon Press.)

This book is written on rather unusual lines. Angles of any magnitude are contemplated from the first, and the proofs of the general theorems relating to supplements and complements of angles and the addition theorems are applicable to all such angles. These proofs are based on cartesian coordinates of points on a circle of unit radius, and will be found of great interest. The book is suitable for university students who already have an acquaintance with simple numerical trigonometry and require a certain amount of higher theory and more advanced applications: as the author says, "the standard" is "in the main suitable for the Intermediate Examinations of Universities," though many propositions of a more advanced character (distinguished by an asterisk) are included.

There is a remarkable slip on p. 16, where the author speaks of "degrees" as *concrete* numbers (Art. 8). This article should be either expunged or rewritten.

The book is full of incidental theorems and ideas of great interest, and the collections of examples are extremely good, particularly the solid problems in the last chapter.

The articles on the triple angle formulae (p. 96), solution of cubic equations (p. 98), properties of triangles (pp. 126-133), various interesting approximations to π and $\sqrt{\pi}$ (pp. 159, 162), and the volume of a spherical segment (p. 175) are among a number of things deserving special attention. A good deal of the book might with advantage be worked through by students who are beginning to specialise for mathematical scholarships.

Elements of Plane Trigonometry. By ROBERT E. MORITZ. Pp. xiv + 361 + 91. $2.00. 1911. (Wiley & Sons, New York; Chapman & Hall.)

This is a well-written treatise, on somewhat old lines, including discussions on logarithms and logarithmic tables, the exponential theorem, and trigonometric series, with a good introduction to complex quantities, and a particularly interesting chapter on the hyperbolic functions, suitable for students who, in addition to a working knowledge of practical trigonometry, wish to have some acquaintance with the higher theory.

Like so many American text-books, it is got up in a most attractive manner, regardless of space, making it very easy to read. There are 90 pages of tables, the logarithmic tables being to 5 decimals, and the natural functions to 4 decimals, followed by one page of constants and their logarithms, to 7 decimal places.

Examples are taken from physics, surveying and engineering, navigation and astronomy. A good feature is the way in which students are taught to check their own numerical work, and not to give results to more figures than are warranted by the data.

There is a good table of contents, and an index, so that reference to any desired part is easy. L. A.

Junior Arithmetic. By R. H. CHOPE, M.A. Pp. 391. With Answers. 2s. 6d. 1912. (Univ. Tut. Press.)

There is so much that is good in this adaptation of Workman's *Tutorial Arithmetic* that I think a more detailed criticism than is usually accorded to "Junior" text-books will not be out of place.

The bad part of the book is the theoretical treatment of fractions. The so-called "proofs" are not even "illustrations." This is what one might expect on reading the definition of a fraction on page 101. The author does not seem to be aware of the fact that a fraction as a number has a totally different interpretation to a fraction as a concrete quantity: or even if he does appreciate the definition of a/b given by the equation

$$[a/b] \times b = a,$$

he does not appreciate that this only defines a/b as a number, and that a/b as a mutiplier or divisor or other operator must be further *defined* so that the fundamental laws of arithmetic should not be contradicted.

Thus, readers are asked to *prove* by means of squared paper that $2 \times \frac{1}{2} = 1$. The only "proof" is that the multiplication by $\frac{1}{2}$ *has* to be *defined* so that the law $a \times b = b \times a$ holds.

Hence $2 \times \frac{1}{2} = \frac{1}{2} \times 2 = 1$ by the definition of a fraction. All this "explaining"—otherwise spoon feeding—is deplorable: the old-fashioned text-books, with their Rule I. Do so and so, Rule II. Do this and that, were immeasurably superior to many modern texts with their half-hearted methods and shuffling of the difficulties they pretend to explain.

With regard to minor points, the author is too inclined to elaborate unimportant points such as the derivation of digit: where he states distinctly that 10 is not a digit, but says that boys "count on their fingers"—surely from 1 to 10 not from 0 to 9. A foreshadowing of the treatment of fractions referred to above occurs on page 32, where he says: "If we have to multiply by 5, clearly we may multiply by 10 and then divide by 2." That is, there is no differentiating between an integral multiplier and a multiplier which is a fractional form. This is a splendid opportunity completely missed. Again, whilst he rightly insists on the "what must be added?" method of subtraction, as far I am aware, he never refers to the omission of the subtrahends in division.

The proofs of the H.C.F. rule look very simple, but what boy will understand them? If H.C.F. cannot be left out, the *rule* should be given without proof as a "short cut." The rules of divisibility should include "simple rules for 7, 11, 13." There is one for these as a group, which any boy who understands the ordinary rule for 11 can follow.

Thus for 4,680,312, we have $312 - 680 + 4 = -364$ and 364 is not divisible by 11, but is by 7 and 13. But $1001 - 364$ is the remainder on division by 1001.

i.e. by $7 \times 11 \times 13$. Hence 7 and 13 are factors.

The phrase "7 will divide into 35" is to be deprecated.

It seems strange in a modern school arithmetic to find such an amount of time given to vulgar fractions and so little in comparison to decimals and the metric system.

The section on approximations is not so clear as it might be, especially with regard to "correct to 3 decimal places," as opposed to "correct to 3 significant figures." There is an error here, a very common one, that correct to 1 decimal figure indicates that there is an error less than 0·1, instead of the true meaning that the error is less than 0·05. Such methods lead to the following example, of which I had a great many instances in a recent examination. An answer coming out 0·00036448 was required correct to 3 significant figures: this was frequently argued out thus:—0·00036448 becomes 0·0003645, *i.e.* 0·000365!

There is a satisfactory treatment of the Unitary method as applied to proportion, interest, practice and so on. But I hardly see the value of the chapter on literal symbols. If this is to form part of arithmetic (as it should) it ought to be brought in right from the beginning. Beginners *can* understand it, as I have proved by experience, just as soon as they can understand "explanations" of such things as "fractions."

The book is well got up and has a quite good selection of examples throughout, together with miscellaneous sets and copies of recent examination papers.

J. M. CHILD.

Analytical Geometry, A First Course. By C. O. TUCKEY and W. A. NAYLER. Pp. 367. 5s. 1912. (Bell & Sons.)

The aim and scope of this book (which consists of 367 pages) is thus set forth in the preface: "This book is designed primarily to provide an interesting course of Analytical Geometry for those who are unable to devote a great deal of time to the subject, and especially to meet the requirements of Engineering, Science, and Army students. At the same time it is hoped that the arrangements adopted will not make the book less useful as a first course for Mathematical Specialists." It may be said at once that within the limits proposed to themselves the authors have executed their task excellently. The treatment throughout is "vocational"; the student sees at once the use of any piece of bookwork by an immediate application. Furthermore, the student has to keep his pencil and his mind going simultaneously throughout the book, the former for diagrams and the latter for the calculations appertaining thereunto. In a word, the geometry and the analysis go hand in hand, and there is not too much reliance on either; the one is called upon to aid the other; if a thing is best done by geometry, it is done by geometry, and if by analysis, it is done by analysis. For special commendation the following points may be noted: the ellipse, hyperbola, circle, parabola are introduced on pages 1 to 14 as cases of arithmetical graphs; on page 52, the perpendicular from a point on a line is found by a more geometrical method than usual—which is easier for the beginner; the rule for writing down the general equation to the tangent is explicitly set down on page 31; the early introduction of analysis to investigate loci on page 85 is particularly good; polar coordinates are introduced and applied to concrete problems in Chapter V., and a few elementary properties of the line, circle, lemniscate and limaçon are investigated,—which convinces the student that polar coordinates are of some use; the following sensible advice is given on page 114 with respect to oblique axes: "Oblique axes are very seldom used, except when their use makes no difference. This will be the case when we are concerned with conditions for concurrence of lines, collinearity of points, or division of lines in a given ratio. When the distance between two points or the angle between two lines is concerned the use of oblique axes brings in complicated conditions, and should if possible be avoided"; parameters and parametric methods receive an excellent introduction on page 115. The subject of Solid Geometry receives an easy introductory treatment towards the end of the volume. The application of the methods of the Calculus with respect to gradients at the beginning of the book enables the student to apply parametric methods with ease and potency. Lastly, the subject is not divided up into water-tight compartments.

The following minor defects have to be noted: it seems undesirable to write $x_1+\Delta x$ instead of $x_1+\Delta x_1$; the evaluation of the root of an equation to many places of decimals is probably best done by Horner's method and the location of roots taught by concrete examples and the cultivation of the pupil's instinct rather than by the application of such formal methods as those given on pages 126-129, at any rate when he is beginning the subject; the introduction of "Line-Equations" is not successful,—a more extended treatment should be given, and practice in drawing the envelopes of arithmetical equations given just as in the case of loci,—it is a pity that this important part of the subject should have received such scanty treatment, but it is to be hoped it will be amended in any future edition. The bookwork should have been printed in clear relief from the examples, etc., for revision purposes.

The book can be thoroughly recommended for the purposes stated by the authors in the preface. There is an excellent collection of examples, both practical and theoretical.

WILLIAM P. MILNE.

Versicherungsmathematik. Von HUGO BROGGI. Pp. viii+358. Price 7 m. 1911. (Teubner.)

This is a translation into German, by Professor Bernstein, of a text-book on the mathematics of Insurance by Professor Broggi of the University of Buenos Ayres. The special object of the book is to present the practical problems of life insurance in a form which will be acceptable to the university student of mathematics with a good working knowledge of the calculus. The book is divided into four sections, dealing respectively with the first principles of the theory of

probability, and the theory of the construction of a mortality table for a stationary population; the fundamental problems of the mathematics of life insurance (including annuities and the insurance of invalid lives), the technical aspects of life insurance (premiums, reserves, profits) and the expectation of loss or gain. The second part contains a very clear sketch of the principles by which the effects of immigration, emigrationd an natural increase can be allowed for in deducing a theoretical mortality table from a statistical record of deaths.

At page 56 a confusion between γ and y has produced a series of misfortunes, and the series in question diverges for all values of γ and not only for $\gamma < 4$.

Der mathematische Unterricht an der deutschen Navigationschule. Von Dr. C. SCHILLING and H. MELDAU. Pp. 82. Price 2 m. 1911. (Teubner.)

Most people are aware that the International Commission on Mathematical Education, originating at the International Congress of Mathematicians held in Rome in 1908 on the initiative of Professor David Eugene Smith, aroused special interest in Germany. The present pamphlet is one portion of that voluminous German report, which testifies to the enthusiasm of the band of workers directed by Professor Klein. The reports as a whole deal in an outspoken manner with mathematical education. The easy-going toleration of incompetence, which is the cruel and bad side of English gentleness, finds no counterpart in Germany, where the trashy or antiquated text-book is sternly dealt with. The present paper deals particularly with the system of training for the mercantile service. Perhaps the most interesting feature to us is the account of the effects of the unified system of examinations in vogue. Bitter complaints, it is stated, have been made by experienced teachers of the crushing out of local initiative and local aspirations by the rigidity of the official system. "Once we could try to teach pupils something: now we have to see that they pass their examinations." The authors do not altogether endorse this view: they point out that the examinations have had a stimulating effect upon the weaker schools.

At a time when we are threatened with a uniformization of our own examination system, just as examinations are rapidly improving, these remarks should not be overlooked.

Collected Papers in Physics and Engineering. By JAMES THOMSON. Selected by Sir J. LARMER and JAMES THOMSON. Pp. civ + 484. Price 15s. net. 1912. (Cambridge University Press.)

There is a quality of mind—which might be described as exalted common sense—giving to its possessors the faculty of "hitting the nail on the head." It is sometimes present in people of little education, and its possessors can sometimes give but a poor account of how it is that they invariably seize upon what proves to be the feature which is decisive of the case. This faculty is especially valuable in problems whose complexity and breadth render the mathematical microscope ineffective, and hence it is frequently found among great soldiers and lawyers.

In conjunction with mathematical genius in the mind of Kelvin it accomplished miracles. James Thomson, with less mastery over mathematical weapons, possessed the faculty, as he showed by his papers on the subject of atmospheric currents and Trade Winds.

Guldberg and Mohn and von Bezold have made progress in the application of definite mathematical processes to the dynamics of atmospheric circulation, but it is nevertheless true that Hadley, Ferrel and James Thomson are the three writers, dealing with the subject by means of general reasoning, whose treatment is regarded as essentially sound.

The same faculty of seizing upon the essential element of a mass of complicated detail was exemplified in Professor Thomson's treatment of the centrifugal pump, and the flow of water over a notch. In both these cases subsequent theoretical investigations have confirmed the results obtained by James Thomson by methods for the most part elementary and general.

The present collection of papers will therefore never become obsolete.

A brief biography by the authors relative to Dr. Bottomley is included in the reprint.

Educationally Professor Thomson was a great coiner of new words—Radian and Interface redeem some score of others, most of which have found happy oblivion.

Calcul des Probabilités. By L. BACHELIER. Tome I. Pp. vii+517. 4to. 25 fr. 1912. (Gauthier-Villars.)

M. Bachelier's treatise promises to be of a very exhaustive character, for questions of historical and philosophic interest are reserved for future volumes.

His chief object in the present volume is to concentrate attention on "continuous" probability, in which the expectation, say, of gain or of error is supposed to vary continuously. Thus, after considering in the first five chapters some of the classical problems on the duration of play and on discontinuous probabilities in general, the same problems are reconsidered and extended by the error-function methods. As no references are given, it is not easy, even with Todhunter's history at hand, to find in what cases a problem or its solution is due to the author himself or derived from earlier authority. In the later chapters the author's plan is to refer other problems to an analogous problem stated in terms of play. Thus, part of the mathematical requisites for the study of the migration of noxious insects, weeds, or epidemics * from a centre, are furnished by imagining three players staking on the x, y, and z of co-ordinates.

M. Bachelier finds something fresh to say of the St. Petersburg problem—in which we are asked to find the expectation of a man who is to receive 2^{n-1} pounds if he throws red at the nth throw of a coin *and not before*. The mathematical expectation is easily seen to be indefinitely large: while common sense is said to assert that no sane person would give fifty pounds for the expectation. M. Bachelier observes: 'Le bon sens ne peut être invoqué quand il s'agit de questions délicates. Il ne permet pas de reconnaître si l'aire comprise entre une courbe et son asymptote est finie ou non. . . . Le joueur doit reçevoir à chaque partie ce qu'il eût reçu la partie précédente multiplié par deux: s'il devait recevoir la même somme multiplié par 1·999 son espérance serait finie: le bon sens ne fait cependant aucune différence entre les deux cas."

At p. 427 there is a slight misprint, $\frac{1}{2}a^2t$ for $\frac{1}{2}a^2t^2$, and the problem solved does not appear to me quite to correspond with the problem stated.

The work, when completed, should provide a repertory of every type of problem in probability.

Wahrscheinlichkeitsrechnung. By A. A. MARKOFF. Translated from the Russian by HEINRICH LIEBMANN. 8vo. Pp. vii+317. 12 marks. 1912. (Teubner.)

Professor Markoff's work is a treatise on probability in general, but its special interest to English readers will probably be that it contains an account not readily accessible elsewhere; of the researches of Russian mathematicians—Tchebycheff, Liapounoff, and Professor Markoff himself—on various problems of probability, particularly those associated with mean values.

The special purpose of these researches is usually the investigation of the limits of error of approximate formulae, and the extension to certain classes of aggregates whose elements are interdependent of theorems relating to the mean values of independent aggregates. C. S. J.

Geometrical Optics. By A. S. PERCIVAL. Pp. ii+132. 4s. 6d. net. 1912. (Longmans, Green & Co.)

This volume is intended to cover the requirements of medical students for their preliminary scientific examinations, but there is no reason why it should not be equally useful to the student of physics. It also contains, according to the author, almost all the Optics required by an ophthalmic surgeon. "As an introduction to mathematical analysis the subject of Geometrical Optics has no equal, for it insists on the importance of paying due attention to the meaning of algebraic signs, and it is also an easy introduction to several somewhat difficult mathematical conceptions. For instance, the vectorial significance of the line BA being equal to $-AB$, or AB taken in the reverse direction, opens up a new vista to the student of Euclid and elementary geometrical methods: equally novel is the conception of a virtual image. At the same time every student can verify for himself the results of his calculations so simply by experiments that it will convince him of the reality of the analytical methods employed." The book seems to be admirably adapted to its purpose. Special attention may be drawn to the treatment of

* Brownlee, *Proc. R.S. Edin.*, 31. 262, and references there.

Cardinal Points, and to Professor Sampson's graphic method on pp. 105-107. We may note *en passant*, that Mr. Percival, in the *Proceedings of the Optical Convention* last year, describes an interesting method of tracing Caustic Curves.

Magnetism and Electricity. A Manual for Advanced Classes. By E. E. BROOKS and A. W. POYSER. Pp. viii+633. 7s. 6d. net. 1912. (Longmans, Green & Co.)

This volume replaces Poyser's *Advanced Electricity and Magnetism*, originally published in 1892. It hardly comes within the province of the *Gazette*, but none the less we venture to recommend it to the attention of our readers as an admirable volume, based almost entirely upon experiment, and brought up to date. It is notable for the insistence upon Faraday's ideas, for the prominence given to the conception of the electric current as the flow of electrons, and for the success with which the authors have accomplished their entirely laudable aim of investing mathematical expressions "with a tangible physical meaning." Each chapter is brought to a close with a considerable selection of questions bearing on its subject matter.

We hope we have said enough to induce such of our readers who are also teachers of science to look at the book, and form their own opinion of its merits. It is certainly the work of teachers of skill and experience.

Annuaire pour l'an 1913, publié par le Bureau des Longitudes. Avec des Notices scientifiques. Pp. vi+707+28+52+16+25+55. 1 fr. 50 c. net. 1912. (Gauthier-Villars.)

This hardy annual contains full tables relative to geography, statistics, metrology, money, and meteorology. For tables of chemical and physical data we must depend on the *Annuaire* for last year or wait till next. The astronomical sections include tables relative to deviation from the vertical in France, varying weight, calculation of altitudes by barometric observations, stellar parallaxes, double stars, proper motions, and stellar spectroscopy. For other tables on seismology, solar physics, minor planets, etc., we must refer to 1912 or the volume for 1914.

The notices referred to in the title are by M. Bigourdan and M. le Commandant Ferrié. They deal respectively with observations on the eclipse of the sun, April 17th, 1912, and the application of wireless telegraphy to the transmission of time to various centres. The orations delivered at the funerals of MM. Radau and Poincaré complete the Notices. There is a full index. We may remind our readers that the first number of the *Annuaire* appeared in 1796.

Key to Prof. W. W. Johnson's Differential Equations. By T. A. LE MESSURIER. Pp. 125. 1·75 8. 1911. (Wiley; Chapman & Hall.)

There is little doubt that the above will be found of considerable use to the student who has to tackle his Differential Equations unaided by the advice of a teacher. Even when he has solved correctly the set of examples at the end of a chapter he cannot be sure that after all he has done them in the shortest and most elegant manner, or that his solutions are complete. Such a key as this if wisely used can do no harm, and in many cases will encourage a private student to persist where without such aid he might be daunted even at the outset. The solutions are clear and concise.

Memoranda Mathematica. A Synopsis of Facts, Formulae, and Methods in Elementary Mathematics. By W. P. WORKMAN. Pp. v+272. With **Five-figure Logarithmic and Trigonometrical Tables.** Arranged by W. E. PATERSON. Pp. 28. 5s. net. 1912. (Clarendon Press.)

This will be found an extremely useful volume of reference for mathematical scholarship candidates and first year men reading for Honours at the University, as it covers almost all the ground for such students. Special attention may be drawn to the admirable manner in which "formulae with their deductions and applications are presented . . . enabling a reader to perceive analogies which he would possibly miss if dependent solely on text-books." Differences of type distinguish between the more and less important sections; notes, shortened proofs, and cross references are to be found in plenty, and there is a full index. With Mr. Workman's skill and experience as a teacher at his back, the student who uses this for revision or other purposes will find himself in safe hands.

Solutions of the Exercises in Godfrey and Siddon's Geometry. By C. L. BEAVEN, M.A. Pp. 164. 5s. net. 1912. (Camb. Univ. Press.)

Mr. Beaven's solutions may be recommended to all in whose hands a key may be lawfully placed. They are as simple as may be, are clearly stated, and a special word of praise may be given to the careful manner in which the diagrams necessary for the questions in Chapter XV. are drawn.

An Elementary Treatise on Statics. By S. L. LONEY. Pp. viii+393. 12s. 1912. (Camb. Univ. Press.)

This treatise will be found useful for scholarship candidates, and it covers the course required for Science and Engineering degrees and for junior students for Mathematical Honours. The reader is assumed to have some knowledge of the methods of the calculus and elementary solid geometry. Sections are devoted to Shearing Stresses, Forces in Three Dimensions, Wrenches, Nul Lines and Planes, Strings and Chains, Attractions and Potential, and Slightly Elastic Beams. It contains a large collection of examples of every degree of difficulty. The quality of Prof. Loney's work is too well known to need further comment. This volume is intended to be a companion to the author's *Dynamics of a Particle and of Rigid Bodies*. It is sure to find an audience, for there are very few text-books covering precisely the same ground from which the student can make his selection.

A Treatise on Plane Trigonometry. By E. W. HOBSON. 3rd Edition. Pp. xv+383. 12s. net. 1912. (Camb. Univ. Press.)

There was a time when Prof. Hobson's volume was the only British text-book in which the higher portions of the subject were adequately treated. Even now it has no serious rival, for although later works on higher analysis contain many chapters overlapping the ground covered by the latter half of this volume, we believe that "Hobson" will continue to be the favourite text-book for those who wish to carry the subject beyond a merely elementary course. Among the many alterations made in this new edition we may note the following: a long section on the length of a circular arc with another on the area of the sector of a circle is introduced into the first chapter. Ferrers' method for the factorisation of $x^n+2\cos n\theta+x^{-n}$ is added to Chap. VII., and by obvious deductions the values of $(x^n\pm1)$, $\sin n\theta$, $\cos n\theta$ are exhibited. The limits of

$$\left(\cos\frac{\theta}{n}\right)^{f(n)} \quad \text{for} \quad f(n)=n,\ n^2,\ n^p \quad (p>2)$$

are discussed, and a more rigorous investigation is given of the series for sine and cosine of an angle in powers of its circular measure. A section is added to the page dealing with De Moivre's Theorem, for the case of the irrational index. Something like a dozen new sections have been added to the chapters on Infinite Series and the Exponential Function. A short account is given of the famous problem known as "squaring the circle," with a modified form of Gordan's proof of the transcendental nature of π. To the chapter on Hyperbolic Functions is added a couple of pages investigating the area of a sector of a rectangular hyperbola. That on Infinite Products is expanded by about ten pages. It will be seen that the additions are substantial. We have noticed a few points that may deserve the attention of the reviser: p. 44, l. 9, for "$\cos(B-C)$" read "$\sin(B-C)$"; p. 77, No. 31, the result should be "$\sin\alpha+\sin\beta=\sin 2\theta$"; p. 79, l. 13, the first bracket should be "$3\alpha+\beta+\gamma$"; p. 187, No. 46, interchange "h, k" in lines 2 and 3; p. 222, No. 83, l. 4, for "P" read "P, Q"; p. 360, No. 17, for "$3n$" read "3^3n," for "$5n^2$" read "5^3n^2," etc.; p. 382, No. 28 also appears as No. 29, p. 123.

An Elementary Treatise on Coordinate Geometry of Three Dimensions. Pp. xviii+381. 10s. net. 1912. (Macmillan.)

The fact that Mr. Bell's treatise on the *Analytical Geometry of Three Dimensions* has already reached a second edition is sufficient proof that its admirable qualities have been widely recognised. The main alterations are of small details in the early part of the book. "One or two sections have been re-written and additional figures and illustrative examples have been added."

The Collected Mathematical Papers of James Joseph Sylvester, F.R.S. Vol. IV. (1882-1897). Pp. xxxvii + 756. 18s. net. 1912. (Cambridge University Press.)

With this volume Dr. Baker brings to a close his arduous and successful labours as Editor of Sylvester's papers. It is prefaced by a delightful biographical notice which is in every respect worthy of the occasion. There is also a reproduction of the portrait which is so familiar an object to those who sit in the Hall of St. John's, and an engraving of the obverse of the R. S. Sylvester medal. There are two indexes—a general index to the four volumes of collected papers and an index to Sylvester's contributions to the mathematical columns of the *Educational Times*. We cannot refrain from quoting a passage from one of Salmon's letters to Sylvester, which hits off to a nicety the habit of the latter, so provoking to his contemporaries, of publishing "imperfect sketches," "written in the very heat of discovery." "I wish you would seriously consider whether it is not a duty everyone owes to society, when one brings a child into the world, to look to the decent rearing of it. I must say you have, to a reprehensible degree, a cuckoo-like fashion of dropping eggs and not seeming to care what becomes of them. Your procreative instincts ought to be more evenly balanced by such instincts as would inspire great care of your offspring and more attention to providing for them in life, and producing them to the world in a more presentable form. Hoping you will meditate on this homily and be the better for it. . . ."

Some idea of the work entailed in editing these papers may be obtained from the two pages of corrections supplied by Mr. Morgan Jenkins to the great paper on the Constructive Theory of Partitions—27 corrections for the 80½ pages. Here also we find in 210 pages the American lectures on Reciprocants—delivered at the age of seventy. How well one remembers what a peep into wonderland was the Inaugural Lecture at Oxford on the Method of Reciprocants, which appeared in *Nature* in 1886! Among the multitude of Notes, etc., we notice that "On a Proposed Addition to the Vocabulary of Ordinary Arithmetic" (manifoldness, totient, Fermatian, etc.), in which he says in a footnote: "Perhaps I may without immodesty lay claim to the appellation of the Mathematical Adam, as I believe I have given more names (passed into general circulation) to the creatures of the mathematical reason, than all the other mathematicians of the age combined." Anything from Sylvester's graceful pen is sure to contain passages that challenge the attention. Here are some: "It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all." "It is usually in the form of some apparent contradiction or paradox that a scientific advance makes its first appearance." "We are thus taught the lesson that all *a priori* reasoning, until submitted to the test of experience, is liable to be fallacious, and it is impossible to prove that a proof may not be erroneous by any other method than that of actual trial of the results which it is supposed to yield." "The Resultant of F, G, H may be compared to a cradle just large enough to contain the twin forms in question (involutants), so as to give assurance that no other form is mixed up with them; and the proof given above shows that this must be the case if neither twin is doubled up upon itself, and if the two do not grow into one another, but like such creatures each possesses a perfectly distinct organisation." "Reading with great pleasure and profit Mr. Forsyth's masterly treatise on Differential Equations (in my opinion the best written mathematical book extant in the English language)." "Wallis, who, like Leibniz, drove three abreast to the temple of fame—being eminent as a theologian, and as a philologer, in addition to being illustrious as the discoverer of the theorem connected with the quadrature of the circle named after him, with which every school-boy is supposed to be familiar, and as the author of the Arithmetica Infinitorum, the precursor of Newton's Fluxions." "In mathematical research, reversing the axiom of Euclid, and converting the proposition of Hesiod, it is a continual matter of experience, as I have found myself over and over again, that the whole is less than its part." "I hope that I may . . . by example, give lessons in the difficult art of mathematical thinking and reasoning—how to follow out familiar suggestions of analogy till they broaden and deepen into a fertilising stream of thought—how to discover errors and to repair them, guided by faith in the existence and unity of that intellectual world which exists within

us, and is at least as real as that with which we are environed." "I do not regard mathematics as a science purely of calculation, but one of ideas, and as the embodiment of a philosophy. . . . Faraday, at the end of his experimental lectures, was accustomed to say—I have myself heard him do so—'We will now leave that to the calculators.' So long as we are content to be regarded as mere calculators we shall be . . . Pariahs." One of the remarks above is repeated in the form: "This is another example of the law in Heuristic that the whole is easier of deglutition than its part." "One is surprised to reflect on the change which has come over the face of Algebra in the last quarter of a century. It is now possible to enlarge to an almost unlimited extent on any branch of it. These thirty lectures . . . might be compared to an unfinished epic in thirty cantos. Does it not seem as if Algebra had attained to the character of a fine art, in which the workman has a free hand to develop his conceptions as in a musical theme or a subject for painting? Formerly it consisted almost exclusively of detached theorems, but now-a-days it has reached a point to which every properly developed algebraical composition, like a skilful landscape, is expected to suggest the notion of an infinite distance lying beyond the limits of the canvas." "I am not one of those who look upon analysis as only valuable for the positive results to which it leads, and who regard proofs as almost a superfluity, thinking it sufficient that mathematical formulae should be obtained, no matter how, and duly entered on a register. . . . It is scarcely possible that a well-reasoned mathematical proof shall not contain within itself subordinate theorems—germs of thought of intrinsic value and capable of extended application." "Thus a theorem of pure form is brought to depend on considerations of greater and less, or as we may express it, Quality is made to stoop its neck to the levelling yoke of Quantity." These random quotations may perhaps send some of our younger members to the pages of the four volumes. Even if the memoirs are over the heads of a reader, he will find purple patches and every now and then catch a glimpse of the matchless fire and inspiring enthusiasm of this great personality. We cannot conclude, however, without quoting the final passages of Mr. Baker's tribute: "He was, however, before all an abstract thinker, his admiration was ever for intellectual triumphs, his constant worship was of the things of the mind. This it was which seems to have impressed those who knew him personally. And because of this, his work will endure, according to its value—mingling with the stream fed by the toil of innumerable men—of which the issue is as the source. He is of those to whom it is given to renew in us the sanity which is called faith."

Practical Geometry and Graphics. By D. A. Low. Pp. vii + 448. 7s. 6d. net. 1912. (Longmans, Green & Co.)

Prof. Low's volume may be regarded as an excellent companion to a course of instruction on these subjects. One of the features of the book is the illustrations, which are of the best, and excellently printed. It is a pity that the volume is not furnished with a tabulated list of sections suitable for a first reading. "Starring" the sections required would attain the same end. A book so packed with matter is apt to make the private student despair of ever reaching the end of his labours. Another of the pitfalls in wait for him is the feeling that he ought to work out all the examples, and it does not appear to be clearly stated anywhere that he is free to select. The book fairly bristles with questions (of which by far the greater number are original); the essence of the training is that each should be worked out on the drawing board; and each takes *some* time. Again, on the fourth page we find the student has to be reminded that the sine of an angle is the cosine of its complement, and so on. By the time he reaches p. 157 or thereabouts he is supposed to be able to tackle the analytical work comprised in the treatment of periodic motion. No doubt the author's experience with technical students has shown him that there is no practical difficulty in this arrangement of the work. And it must be admitted that the book is explicitly intended for the technical student. But from the point of view of the unaided student who is getting his knowledge of the subject from the book, it is a blemish that might be removed by a few lines of indication as to the complementary mathematical work that should be done in the period intervening between beginning the subject and reaching p. 157. This is in our opinion the only adverse reflection that can be made. Prof. Low's expository power is well known, and is here to be seen at his best. No

student who is able to go through this volume with the aid of a competent instructor can fail to get a thorough grasp of the principles so clearly set forth. The book is beautifully got up and is a pleasure to read. The leaded headings of the sections and the index make the work of reference easy. The usual apparatus of mathematical tables is supplied in an appendix.

CORRESPONDENCE.

THE UNIVERSITY OF CHICAGO,
Founded by John D. Rockefeller,
The Board of Recommendations, *January*, 1913.

Office of the Secretary.

DEAR SIR,

The National Committee on Geometry Syllabus, working under the joint auspices of the American Federation of Science and Mathematical Teachers and the National Education Association, has just completed its final report, and has made arrangements with the Commissioner of Education for its distribution to teachers of mathematics and others interested. Will you kindly insert the following statement in one or more issues of your journal, in order that your readers may have the privilege of securing copies.

Yours very truly, H. E. SLAUGHT,
Chairman of the Committee.

The report of the National Committee of Fifteen on Geometry Syllabus, which has been under consideration for nearly three years, and which was revised and finally adopted at the N.E.A. meetings in July, 1912, has now been re-published in a pamphlet of 80 pages, and is ready for distribution to teachers of geometry and all others interested. This report was prepared under the joint auspices of the American Federation of Teachers of the Mathematical and Natural Sciences and the National Education Association. It includes a historical introduction and sections on axioms and definitions, on exercises and problems, and the syllabus itself, including both plane and solid geometry. It is the hope of the committee that this report may be of great service to all teachers of geometry, and to this end that it may have a wide distribution among all interested. Copies may be secured gratis upon application to the Commissioner of Education, Department of the Interior, Washington, D.C.

THE LIBRARY.

THE Library has now a home in the rooms of the Teachers' Guild, 74 Gower Street, W.C. A catalogue has been issued to members containing the list of books, etc., belonging to the Association and the regulations under which they may be inspected or borrowed.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Wanted by purchase or exchange:

1 or 2	copies of *Gazette*	No. 2 (very important).
2 or 3	copies of Annual Report	No. 11 (very important).
1 or 2	„ „	Nos. 10, 12 (very important).
1 copy	„	Nos. 1, 2.

ERRATA.

l. 8, p. 70, "it" *after* "meeting."
l. 14, *for* $a:b$ *read* $b:a$.
Index, Vol. V. p. vi, l. 20, *for* "W. Gallatly" *read* "Cecil Hawkins."
l. 28, p. 339, *for* $2^{83}-1$ *read* $2^{89}-1$.

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